15

A Mathematical Look at Vehicle Dynamics

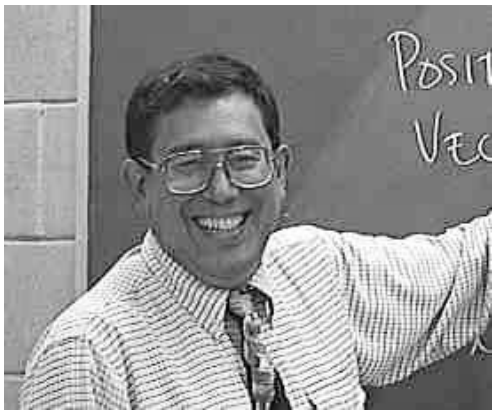
Dr. J. J. Salinas and W. J. Rowley

With Second Printing Additions by **Wm. C. Mitchell**

(It's just not this simple!)

About Juan Salinas

Dr. Juan Salinas – Juan is a brilliant engineer, mathematician and engineering professor (his office walls are covered with teaching awards). He is also one of the **most** modest and nicest people I have ever met. He is a true renaissance man whose range of experiences and expertise continually dumbfound me – from nuclear power plants to satellite global positioning to consulting for the United Nations. Juan has a **great** mixture of theoretical and hands-on experience.



I first met Juan when he was doing postdoctoral work in engineering while I was doing research into brain sense stimulation and deprivation (controlling three dimensional environments through the use of various brain frequencies). Juan often volunteered as a guinea pig for some of the more bizarre experiments. However, after one particularly outlandish test, we had great difficulty coaxing Juan back to the real world. Some say he has never been the same since, though I like to think he just got better! Sorry about that, Juan.

Juan has been **instrumental** in turning my basic theories and formulas into the refined matrix driven models found in the dynamics chapters. It seems that no matter how complex the problems I find for Juan to solve, he always exceeds my expectations. The **unpaid** hours that he has spent on this project are quite astronomical. Juan, I have **no** idea how I will ever repay you.

Juan, for reasons I do not fully understand, will **only** work on race related projects through the umbrella of Rowley Race Dynamics so if your team has a particularly difficult engineering problem you would like Dr. Salinas to solve, please contact me at rowleyrace@telus.net and I will make the appropriate arrangements.

Important Note: Since the first printing, Wm. C. Mitchell and the author have spent well in excess of 850 hours further refining the Wan31 models. We had first intended to only spend one or two hours realigning several cells that had become corrupted during the first printing. However, as we were already developing a similar but far more complex model (based on Dr. Salinas's mathematical model and Wm. C. Mitchell's WinGeo3 software) for a number of teams, we decided we would also include several of these new developments in the Wan31 models.

Not All Great Minds Think Alike: Though both Juan Salinas and Bill Mitchell agree that the Wan31 model is working correctly there are still small components within the Chapter 15 mathematical analysis with which they don't agree (not the outcome, just the process)? As the author, I agree with both of them.

Modeling the Behaviour of Race Car Suspension Components

J.J. Salinas

Introduction

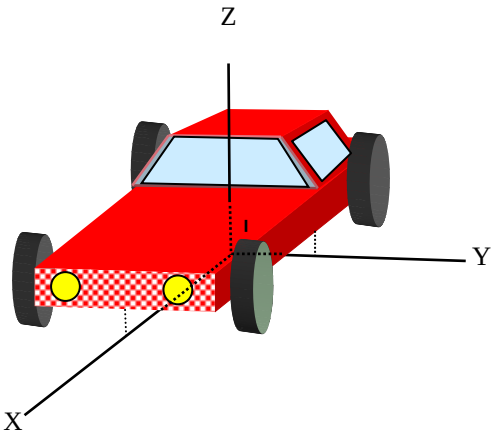
The motion of a racing car consists of translation and rotation. If we describe this motion using a three-dimensional, co-ordinate system with axes X, Y and Z, the translation may be described in terms of three displacements in the direction of the axes. The rotation may also be described in terms of three component rotations about the co-ordinate axes. We can say then that the car, as a whole, has six degrees of freedom (DOF), three translations and three rotations.

The three translations are:

- motion in the direction of travel (X–Axis)
- sideways motion (Y–Axis)
- vertical motion (Z–Axis)

The three rotations are

- roll (About the X–Axis)
- pitch (About the Y–Axis)
- yaw (About the Z–Axis)



The components of the car are held together by rigid or flexible connections such as bars, struts, links, springs, roll bars or dampers. The individual behaviour of these components plays an important role in explaining the overall behaviour of the car. The interactions between the wheel assemblies and the chassis, through the connections, rigid and flexible, are complex and difficult to assess in detail.

Various authors have proposed simplifications in the model parameters to permit a reasonable evaluation of the behaviour of the car and its various components. These simplifications usually consist of reducing the 3-dimensional system to a 2-dimensional model. One such simplification is the ‘bicycle model’ where the properties of the two front wheels are combined to form one single wheel and a similar operation is performed on the rear wheels. The resulting model then consists of a front, steering wheel and a rear, propulsive wheel, just like a bicycle.

To overcome the limitations of the bicycle model, the ‘wheel pair model’ was developed. In this model, the effects of loads and other demands on the system are split into ‘front’ and ‘rear’ components and each “wheel pair’ is then modeled independently.

The next step from these two-dimensional models is to utilize a 3D analytical model. Visualization of translation and rotation components in a complex system can be simplified if the 3D problem is examined as a combination of three 2D problems. For this reason, we will define 3D model parameters using 2D plan, side and elevation views of the system and its major components.

Parameters

To explain the behaviour of a racing car, at rest or in motion, we need to consider a number of input and output parameters. Input parameters are known or measured dimensions, position co-ordinates, spring rates, loads, etc. Output parameters consist of loads and deformations or displacements. The analytical model derived here uses basic principles of mechanics, strength of materials, linear algebra and analytical geometry to develop mathematical expressions to characterize the relationships between input and output parameters.

The basis of the 3D model is the assumption that wheel assemblies have only one degree of freedom. That is, wheels can only move up and down, and in doing so they activate the rigid and flexible suspension components. Loads on the car consist of its own weight and that of other components such as ballast, driver and fuel plus the kinetic effects

of turning, acceleration and braking, in addition to aerodynamic loads. These loads are applied to the chassis and passed through the suspension components to the wheel assemblies and to the ground.

The handling characteristics of the car are dictated, to a large degree, by the vertical forces at the contact patch of the four tires. Finding the value of these contact patch forces under the effect of the various loads and geometric and material constraints is the main focus of the analysis.

The motion characteristics of the chassis are defined in terms of an overall vertical displacement (squat), a lateral rotation (roll) and a longitudinal rotation (pitch). We will not consider the yaw motion component. Squat, pitch and roll are calculated in terms of the vertical displacements at each corner. Finding the value of these four vertical displacements is the second part of the main focus of the analysis.

Input parameters consist of loads, dimensions and material properties (*e.g.*, spring rates). Output parameters consist of 4 forces and 4 displacements.

Components

The system components are: the chassis, the wheel assemblies and the suspension. The chassis is assumed to be a rigid body, supported by the suspension components and having three degrees of freedom: a vertical displacement (squat), a lateral rotation (roll) and a longitudinal rotation (pitch). The wheel assemblies are also rigid bodies but have only one DOF, namely the vertical motion of the wheel. The suspension components can be *rigid* or *flexible*. Rigid suspension components can be links, struts or track bars. Flexible suspension components can be springs, roll bars or dampers. Flexible suspension components have one DOF. Rigid components have none.

Suspension Systems

Two types of suspension systems will be considered, *independent* and *solid axle*. Independent suspension systems consist of (rigid) bars or links and (flexible) springs, roll bars and dampers. They can be used in the front corners or in the rear.

Solid axle suspension systems, as the name implies, consist of a single (rigid) axle joining two wheel assemblies plus one (rigid) track bar connecting the axle to the chassis and (flexible) springs, roll bars and dampers at each corner. For the purpose of this book, solid (live) axle suspensions are used only at the rear.

Co-ordinate system

To make reference to the location of components and loads we define a co-ordinate system making the following assumptions:

1) The origin is an arbitrary point located at or near the geometric centre of the plan view of the car, at ground level.

2) The X-axis runs along the length of the car, at ground level. Positive sense is towards the front of the car.

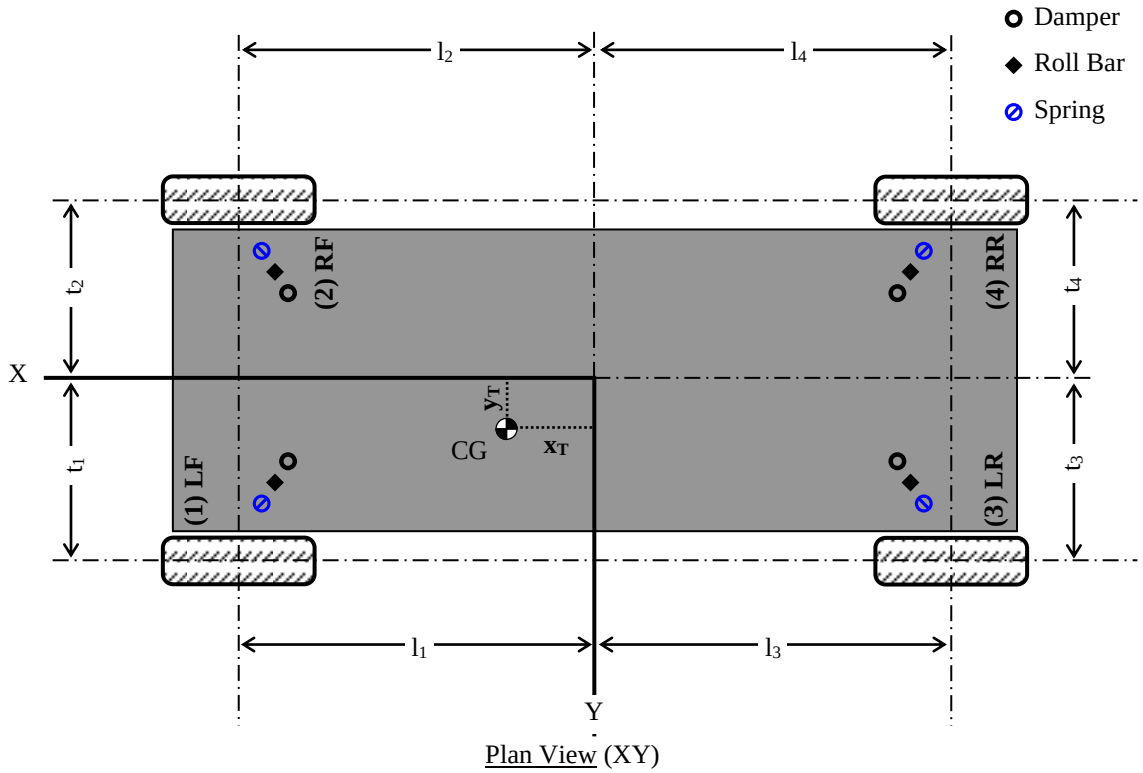
3) The Y-axis runs parallel to the wheel axes, at ground level. Positive sense is towards the left side of the car.

4) The Z-axis is vertical. Positive sense is upwards.

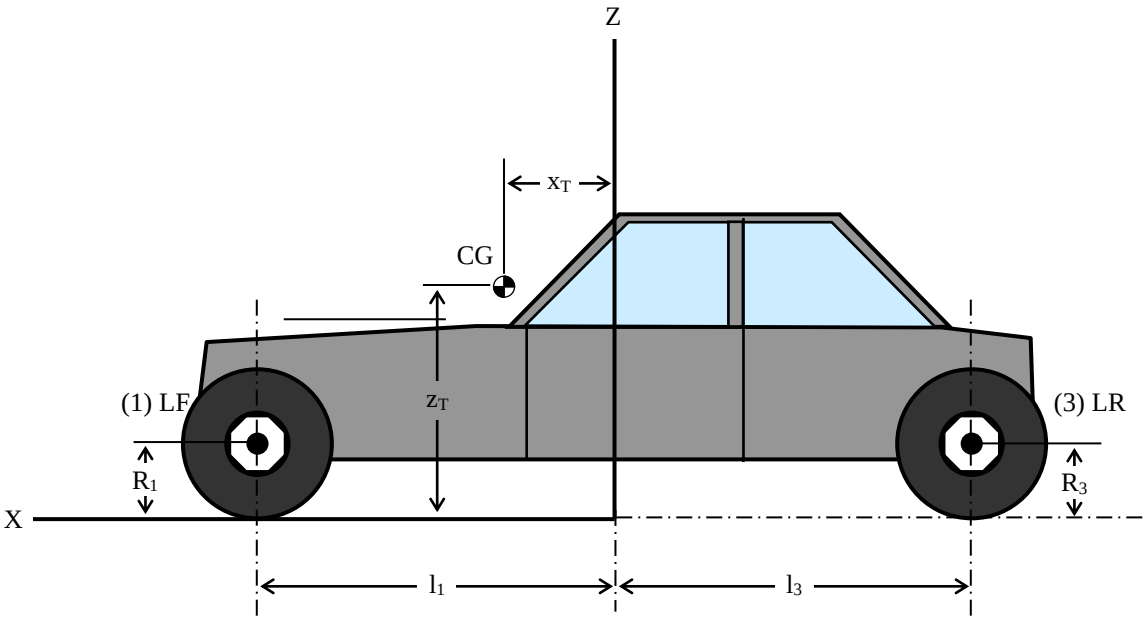
5) The system must be “right-handed”. This means that if we place the heel of the right hand on the X-axis while the fingers curve towards the Y-axis, the thumb points in the direction of the Z-axis.

Geometry

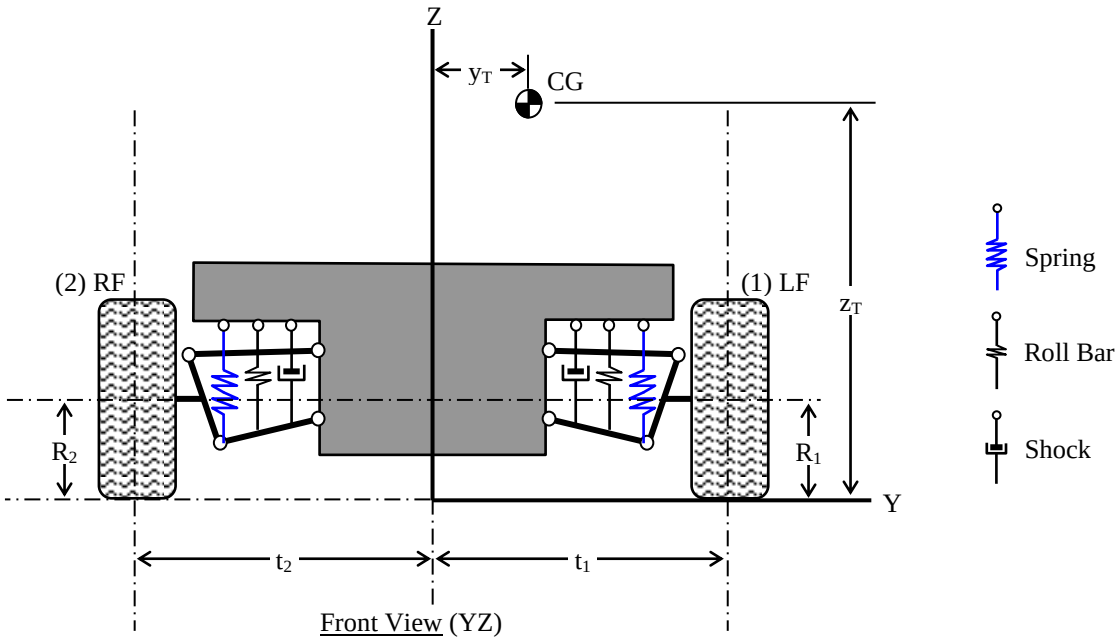
The analysis considers a completely asymmetric configuration for all the components. This means that it is possible to have different values for half-track, wheelbase, wheel size, and spring/roll-bar/damper location at all corners. This includes the possibility of also having different values for spring rates and motion ratios at all corners. The following figures show the co-ordinate system, basic dimensions and suspension types used in the analysis.



Plan View (XY)

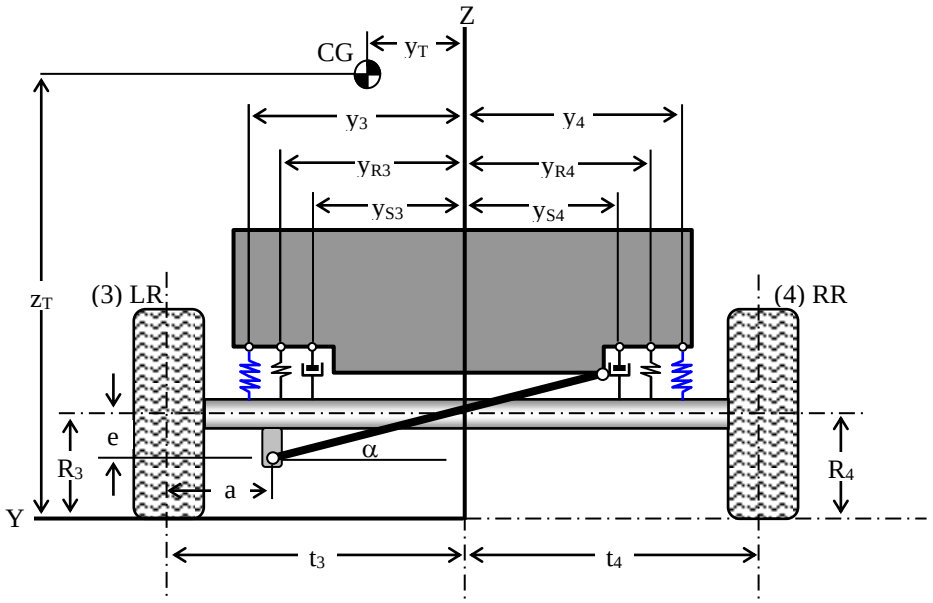


Left Side View (XZ)



Front View (YZ)

Independent Suspension



Rear View (YZ)

Solid Axle. Track Bar

Notation

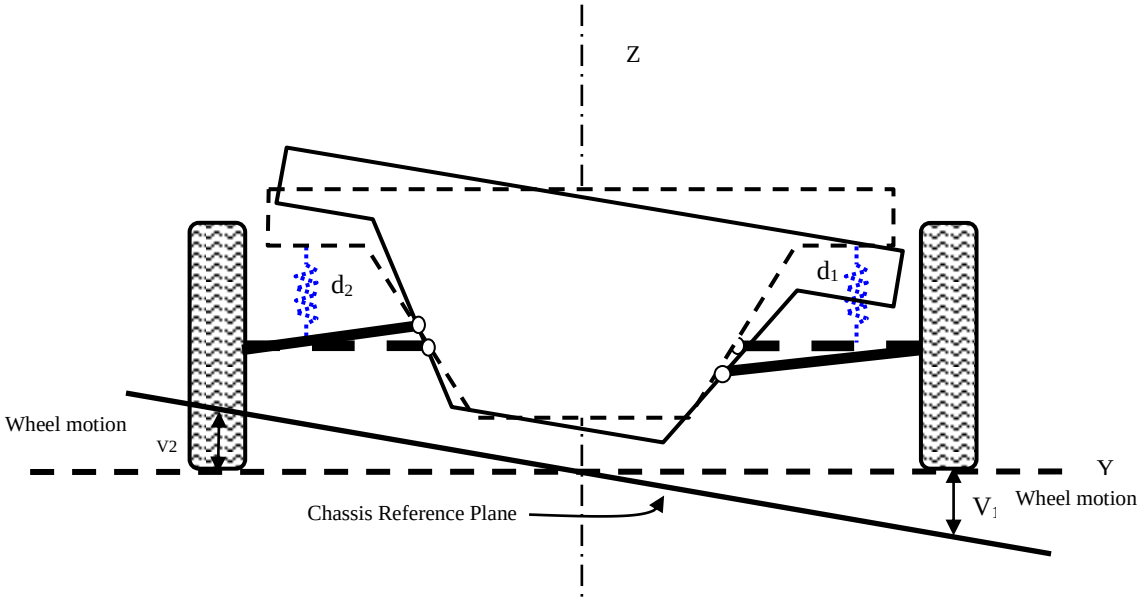
(1) LF	Left Front corner
(2) RF	Right Front corner
(3) LR	Left Rear corner
(4) RR	Right Rear corner
ℓ_i	Half wheel base (i = 1 … 4)
t_i	Half track (i = 1 … 4)
V_i	Wheel motion
x_i	Fore-aft Wheel Location +/- ℓ_i Half Wheelbase
y_i	Lateral Wheel Location +/- t_i Half Track
CG	Centre of Gravity for total mass
x_T, y_T, z_T,	CG co-ordinates
R_i	Wheel radius (i = 1 … 4)
e, a, α	Track Bar location and orientation

The Half wheelbase values are denoted by ℓi. These values are fore-aft DISTANCES from the origin and are positive for both front and rear wheels. x_i is the fore-aft location of the wheels and is positive for the front wheels and negative for the rear.

The Half-track values are denoted by ti = Half Track. These values are DISTANCES from the origin centerline and are positive for both left and right wheels. yi is the lateral location of the wheels and is positive for the left wheels and negative for the right wheels.

Wheel Motion

The chassis is represented by a rigid platform while the tire contact patch points lie on a reference plane normally called the ground plane. When the chassis moves, this reference plane moves with the chassis as if it were attached. The suspension and tires move so that the tires remain in contact with the ground. Wheel motion, sometimes called wheel travel, is the vertical distance from the tire contact point on the ground to the reference plane. As the chassis moves vertically (heaves or squats), rolls or pitches, the distances between the ground and the reference plane changes. This is wheel motion. Wheel motion can change as the result of the chassis moving, one or more wheels moving (bump) and/or any combination of the two.



If the chassis squats (moves down), the vehicle exhibits POSITIVE wheel motion. The wheels move up in relation to the chassis.

If the chassis rolls to the right, as in a left hand turn, the right side chassis points go down and exhibit POSITIVE wheel motion. The left side chassis points go up and exhibit NEGATIVE wheel motion as the wheels move down in relation to the chassis.

The spring displacements have the same sign as the wheel motions. As the chassis squats, the wheel motions are positive and the springs compress with POSITIVE displacement. When the chassis heaves (rises away from the ground), the wheel motions are negative and the springs extend with a NEGATIVE displacement.

Frame of Reference

"Frame of reference describes the viewer's perspective. In most of this book, we use the 'world' reference wherein the ground is fixed and the chassis moves about above the track. The engineer, standing beside the track, sees the vehicle pitch and roll as it moves along the track. This is the normal viewpoint of authors, software writers and motorsports engineers.

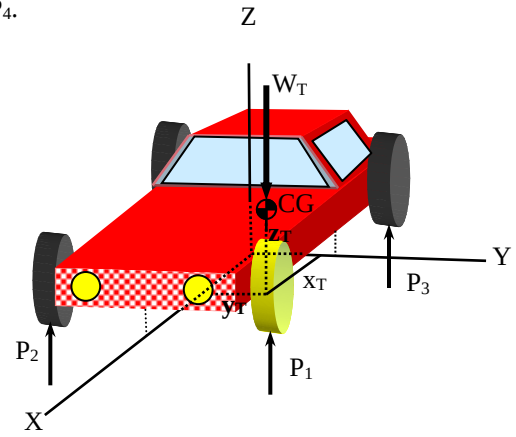
The driver has a different frame of reference, often called the 'vehicle' frame of reference. From this viewpoint, the vehicle remains fixed and the world moves around the vehicle. The scenery rushes by the window and the ground moves under the tires. This frame of reference is shown by in-car video. A wind tunnel exploits the driver's frame of reference. In the earth's frame of reference, the vehicle moves through the air. In the vehicle frame of reference, the vehicle is fixed and the air moves past. This is what makes a wind tunnel useful.

To the driver, the chassis remains fixed to his body and the world revolves around him. Indeed, to most drivers, successful or not, they are the center of the world.

Many things, including wheel motion, are clearer in the driver's frame of reference. In this frame of reference, the wheel motion is simply how far the ground moves under each tire." Wm. C. Mitchell

Centre of Gravity

To determine the x_T and y_T co-ordinates of the centre of gravity, the car is placed on scale pads and the load at each wheel is measured: P_1, P_2, P_3, and P_4.



Static Analysis

We consider the plan view of the car and write the following equilibrium equations: Sum of vertical forces, sum of moments about the X-axis and sum of moments about the Y-axis must equal zero.

$\Sigma F_Z = 0$: [1] $P_1 + P_2 + P_3 + P_4 = W_T$

$\Sigma M_X = 0$ [2] $P_1 t_1 - P_2 t_2 + P_3 t_3 + P_4 t_4 = W_T Y_T$

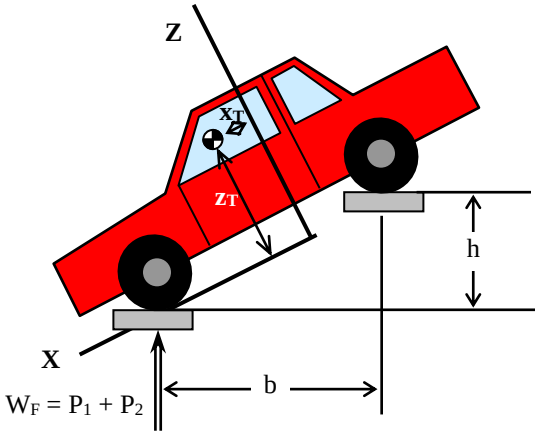
$\Sigma M_Y = 0$ [3] $P_1 l_1 + P_2 l_2 - P_3 l_3 - P_4 l_4 = W_T X_T$

Solving

$$X_T = \frac{P_1 l_1 + P_2 l_2 - P_3 l_3 - P_4 l_4}{W_T}$$ and $$Y_T = \frac{P_1 t_1 - P_2 t_2 + P_3 t_3 - P_4 t_4}{W_T}$$

Elevation of CG (z_T)

The z_T co-ordinate of the CG is determined using a special procedure described elsewhere. In this procedure, the rear end of the car is raised and the scale pad loads on the front wheels are measured for various angles. The co-ordinate Z_T is determined as a function of the front end loads, the angle of elevation, wheel-base dimensions, CG location (X_T,Y_T) and other geometric parameters.



The formula given below is based on the one proposed by Milliken and Milliken (p670).

$$z_T = \frac{W_F(l_1 + l_2 - 2x_T)}{2W_T\,(h/b)} + \frac{(R_1 + R_2)(l_3 + l_4 + 2x_T) + (R_3 + R_4)(l_1 + l_2 - 2x_T)}{2(l_1 + l_2 + l_3 + l_4)}$$

It is recommended to use various settings for h and W_F and to use an average value for z_T. We now have the co-ordinates x_T, y_T, z_T for the CG and will be using them to calculate wheel loads and spring deflections later on.

Bias

Once the location of the CG has been determined (x_T, y_T, z_T), it is possible to adjust the geometry of the car (track, wheel base, etc.) to change the scale pad wheel loads until target values of left bias, rear bias and wedge are achieved. These values of bias and wedge are dictated by the type of car, race standards, specifications or rules, racetrack geometry and other parameters. Values of scale pad wheel loads P_1, P_2, P_3 and P_4 are used to calculate bias and wedge values as follows:

$$\text{Left Bias} = \frac{\text{Left side loads}}{\text{Total Car Load}} \quad \rightarrow \quad B_L = \frac{P_1 + P_3}{W_T}$$

$$\text{Rear Bias} = \frac{\text{Rear loads}}{\text{Total Car Load}} \quad \rightarrow \quad B_R = \frac{P_3 + P_4}{W_T}$$

$$\text{Wedge Bias} = \frac{\text{Rear Left Load} + \text{Right Front Load}}{\text{Total Car Load}} \quad \rightarrow \quad B_w = \frac{P_2 + P_3}{W_T}$$

Equilibrium $\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \quad W_T = P_1 + P_2 + P_3 + P_4$

This system of four equations and four unknowns can be written as follows:

$$\begin{aligned} P_1 + P_2 + P_3 + P_4 &= W_T \\ P_1 + P_3 &= W_T B_L \\ P_3 + P_4 &= W_T B_R \\ P_2 + P_3 &= W_T B_W \end{aligned}$$

In turn, these equations are written in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = W_T \begin{Bmatrix} 1 \\ B_L \\ B_R \\ B_W \end{Bmatrix}$$

Solving

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^{-1} W_T \begin{Bmatrix} 1 \\ B_L \\ B_R \\ B_W \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} W_T \begin{Bmatrix} 1 \\ B_L \\ B_R \\ B_W \end{Bmatrix}$$

To determine scale pad loads required to meet target values of bias and wedge, we can use the following relationships:

$$P_1 = \frac{W_T}{2}\left(1 + B_L - B_R - B_W\right)$$

$$P_2 = \frac{W_T}{2}\left(1 - B_L - B_R + B_W\right)$$

$$P_3 = \frac{W_T}{2}\left(-1 + B_L + B_R + B_W\right)$$

$$P_4 = \frac{W_T}{2}\left(1 - B_L + B_R - B_W\right)$$

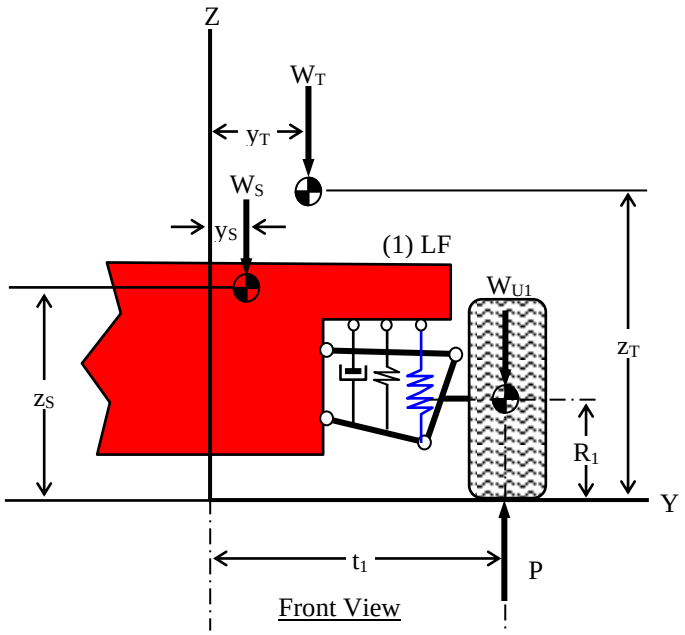
With the formulas derived so far, we can perform three very important calculations:

1) Determine the x_T and y_T co-ordinates of the CG for known scale loads
2) Calculate scale loads for known values of bias and wedge
3) Calculate bias and wedge for known values of scale loads

A distinction is made in the analysis between the *total mass* (W_T) the *sprung mass* (W_S) and the *unsprung mass* (W_U) of the car. Each one of these components has its own centre of gravity and their co-ordinates must also be determined. The following sections discuss how to determine these important quantities.

Unsprung Mass (W_U)

To determine the location of the centre of gravity of the unsprung weight, we assume that we know the weight of the wheel assembly at each corner and assume it to be located at the wheel axle, directly above the middle of the contact patch.



Independent Suspension Left Front Corner

The co-ordinates (x_U, y_U, z_U) of the centre of gravity for the overall unsprung weight are determined in the conventional manner, as we did for the total weight. And to find the elevation of the centre of gravity for the *unsprung* weight requires a special construct, as shown below.

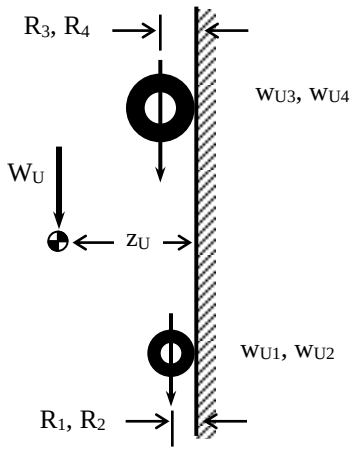
$$W_T = W_U + W_S$$

$$W_U = w_{U1} + w_{U2} + w_{U3} + w_{U4}$$

$$x_U = \frac{w_{U1}l_1 + w_{U2}l_2 - w_{U3}l_3 - w_{U4}l_4}{W_U}$$

$$y_U = \frac{w_{U1}t_1 - w_{U2}t_2 + w_{U3}t_3 - w_{U4}t_4}{W_U}$$

$$z_U = \frac{w_{U1}R_1 + w_{U2}R_2 + w_{U3}R_3 + w_{U4}R_4}{W_U}$$



Sprung Mass (W_S)

To determine the weight of the 'sprung mass' and the location of its centre of gravity, we use basic principles of mechanics, as shown.

$$W_S = W_T - W_U$$

$$x_S = \frac{W_T x_T - W_U x_U}{W_S}$$

$$y_S = \frac{W_T y_T - W_U y_U}{W_S}$$

$$z_S = \frac{W_T z_T - W_U z_U}{W_S}$$

Analysis

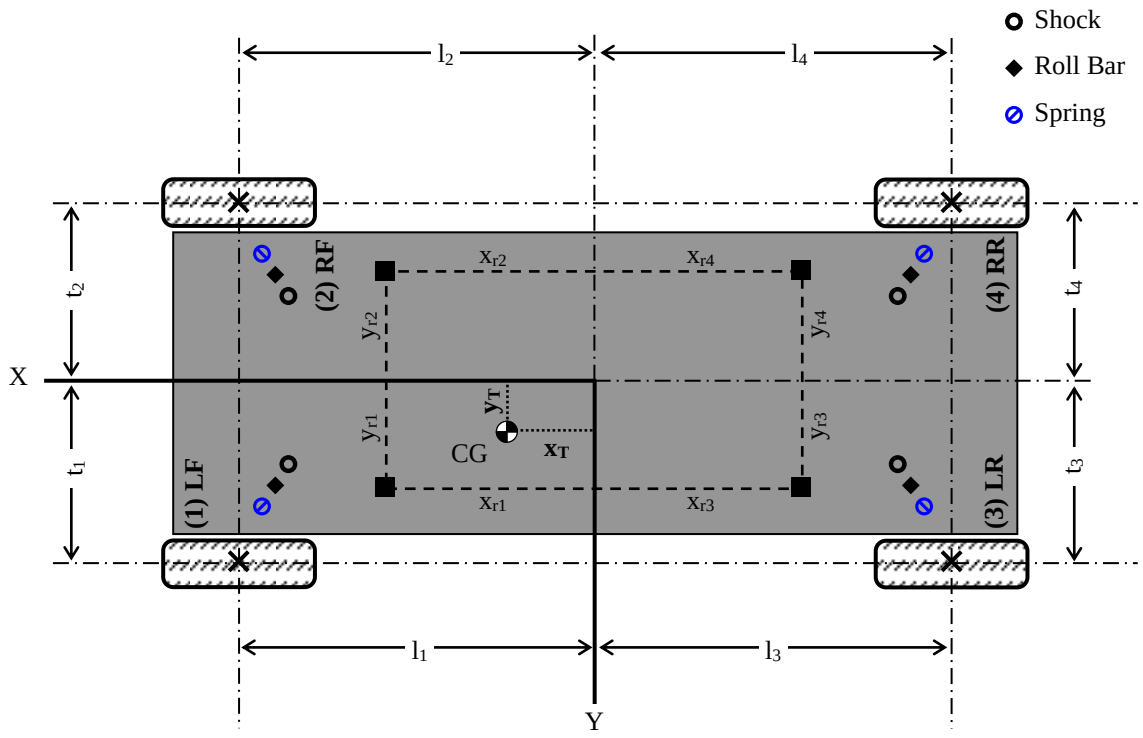
To calculate wheel loads and rides for a known geometry and suspension parameters, we use an analysis procedure considering:

(a) Equilibrium of vertical (Z) forces and moments about horizontal axes (X,Y)

(b) Material behaviour of springs and other suspension components (Hooke's Law)

(c) Compatibility of deformations of the rigid chassis. (Compliance)

At this stage, the equilibrium analysis of forces and moments considers only the vertical force at the contact patch under each tire. Material behaviour considers the relationship between the force and deformation of each spring, roll bar and shock. Compatibility of deformations assumes that the chassis is a rigid body and the displacements, used to determine ride height, must satisfy certain geometric constraints. In brief, we will work with loads at the contact patch, forces and deformations on the suspension components and final displacements at arbitrary points where ride height is measured. We also need to know the location (co-ordinates) of each component. The diagram below shows the location of contact patch and ride points.



Location of contact patch and ride points

Contact Patch Load (P)

Spring Force/Deformation (F/d)

Ride Point (r)

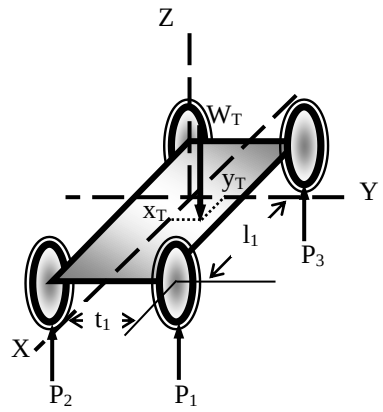
CG

Basic Model

To show the general concepts used in the analysis, we will consider a flat rigid plate supported at the four corners by springs of known stiffness. This simple model will be expanded later to incorporate other suspension components.

Equilibrium

Consider only the effects of vertical loads (parallel to the Z-axis)



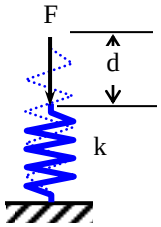
$\Sigma F_z = 0$ [1] $P_1 + P_2 + P_3 + P_4 = W_T$

$\Sigma M_x = 0$ [2] $P_1 t_1 - P_2 t_2 + P_3 t_3 - P_4 t_4 = W_T y_T$

$\Sigma M_y = 0$ [3] $P_1 l_1 + P_2 l_2 - P_3 l_3 - P_4 l_4 = W_T x_T$

Spring Behaviour (Hooke's Law)

Assume linear elastic springs with known spring rate.



[4] $F_1 = k_1 d_1$

[5] $F_2 = k_2 d_2$

[6] $F_3 = k_3 d_3$

[7] $F_4 = k_4 d_4$

where

F = Spring force, lb
d = Spring deflection (+ in compression), inches
k = Spring rate, lb/in

Spring preload

Sometimes it is necessary to adjust the springs to change the ride or the wheel load. This change in spring length must be taken into account in the spring behaviour calculations and equations [4] through [7] now look as follows:

[4] $F_1 = k_1(d_1 + D_1)$

[5] $F_2 = k_2(d_2 + D_2)$

[6] $F_3 = k_3(d_3 + D_3)$

[7] $F_4 = k_4(d_4 + D_4)$

where D is the amount of pre-load in inches.

In order to have an homogeneous system of linear equations in variables P (wheel loads) and d (spring deflections), we need to 'convert' spring loads to wheel loads using the concept of *Motion Ratio*.

Motion Ratio

Race engineers use the concept of Motion Ratio to relate spring deflection to wheel motion.

Spring displacement d = m V - or - wheel motion V = d / m

The same motion ratio relates wheel load to spring force.

Wheel Load $P = m F$ - or - Spring force $F = P / m$

Combining these equations with Hooke's Law of spring behavior gives

Wheel load $P = m F = m k d = m k m V = m^2 k V$

If we define the wheel rate K as $m^2 k$ then $P = K V$

Actual motion ratios require an involved 3D analysis beyond the scope of the concepts presented here. The reader is referred to the computer program WinGeo3 by Wm.C. Mitchell to obtain accurate values of motion ratios, m, for given car suspension configurations.

In addition to the load passed by the spring to the wheel (mF), we must also consider the weight of the wheel assembly (w_{ui}) and add it to this load to determine the total load on the wheel (P). Equations [4] to [7] become

[4] $P_1 = K_1V_1 + m_1k_1D_1 + w_{u1}$

[5] $P_2 = K_2V_2 + m_2k_2D_2 + w_{u2}$

[6] $P_3 = K_3V_3 + m_3k_3D_3 + w_{u3}$

[7] $P_4 = K_4V_4 + m_4k_4D_4 + w_{u4}$

Wheel Radius Change, R_C

A change in the wheel radius (R_C) can be considered to be a prescribed vertical displacement at the contact patch (R_{Ci}). This is passed on to the spring, through the motion ratio, with a resultant change on the wheel load. Equations [4] through [7] now look like this:

[4] $P_1 = K_1 (V_1+R_{C1}) + m_1k_1D_1 + w_{u1}$

[5] $P_2 = K_2 (V_2+R_{C2}) + m_2k_2D_2 + w_{u2}$

[6] $P_3 = K_3 (V_3+R_{C3}) + m_3k_3D_3 + w_{u3}$

[7] $P_4 = K_4 (V_4+R_{C4}) + m_4k_4D_4 + w_{u4}$

Compatibility of Deformations

Equations [1] through [7] involve 8 unknowns: 4 wheel loads (P) and 4 wheel motions (V). We need an eighth equation and to find it we assume that all four wheel motions lie on a plane.

From principles of analytic geometry we know that we only need three arbitrary points in space to define a plane. A fourth point, in the same plane, must obey certain compliance or compatibility conditions. Here the math gets a bit more involved but with knowledge of basic linear algebra, it is possible to arrive at a general formulation for the equation of the plane containing all four wheel motions. The equation of the plane is an analytical relationship between the three co-ordinates (x, y and z) of a point on the plane itself and can be used to calculate the elevation of a point (z) if we know its position (x and y) in a plan view.

We determine the equation of the plane using the location (x,y) and vertical displacement ($z = V$) of the wheel motions. We then use the equation of the plane with the co-ordinates of wheel motions (x_r and y_r) to find the vertical displacement of the chassis (z_r) at those points we have chosen to measure the ride. The following assumptions form the basis for the analysis:

(a) Rigid chassis

(b) All four wheel motions lie on a plane

(c) Ride change at arbitrary points is calculated at the projection of these ride points onto the plane containing the wheel motions.

Linear algebra tells us that the four wheel motions (a point at the contact patch) are on a plane if the determinant shown below is equal to zero.

$$\begin{vmatrix} x_1 & y_1 & V_1 & 1 \\ x_2 & -y_2 & V_2 & 1 \\ -x_3 & y_3 & V_3 & 1 \\ -x_4 & -y_4 & V_4 & 1 \end{vmatrix} = 0$$

Solving for this condition, in terms of wheel motions V_1, V_2, V_3, V_4, we obtain the compatibility equation [8] for the general case of an asymmetric chassis.

[8] $C_1 V_1 + C_2 V_2 + C_3 V_3 + C_4 V_4 = 0$

where

$C_1 = (-x_3 + x_4)\, y_2 + (x_2 + x_4)\, y_3 + (x_2 + x_3)\, y_4$

$C_2 = (x_4 - x_3)\, y_1 - (x_1 + x_4)\, y_3 - (x_1 + x_3)\, y_4$

$C_3 = -(x_2 + x_4)\, y_1 - (x_1 + x_4)\, y_2 - (-x_1 + x_2) y_4$

$C_4 = (x_2 + x_3)\, y_1 + (x_1 + x_3)\, y_2 + (x_1 - x_2)\, y_3$

Constants C_1, C_2, C_3, C_4, are called the 'Compatibility Coefficients' and for a symmetric chassis $C_1 = -C_2 = -C_3 = C_4$. Equation [8] then becomes

$V_1 - V_2 - V_3 + V_4 = 0$

Ride

Ride is measured from the ground to convenient, albeit arbitrary, points on the rigid chassis. These measurements are taken at points not directly coinciding with the position of the wheels. However, the assumption of a rigid chassis makes it possible to determine the ride change at these points if we know the wheel motion at each corner. The procedure consists of using the calculated wheel motion (V_1, V_2, V_3 and V_4) to derive the equation of the plane of the wheel motions. We then substitute the co-ordinates of the points where we wish to find the ride change in the equation of the plane to calculate ride changes.

Vertical Motion analysis

Calculated wheel loads are used to check compliance with target values of *wedge* and *bias*. Calculated wheel motions are used to determine motion components of the ride such as *squat*, *pitch* and *roll*. We assume the chassis to be a rigid body and that the four wheel motions are contained in a plane. If we establish the equation of this plane we can find the elevation of any point on the plane if we know its relative position (*x* and *y* co-ordinates) as seen on a plan view of the chassis.

Equation of the Plane

The plane containing the four wheel motions has an equation of the form:

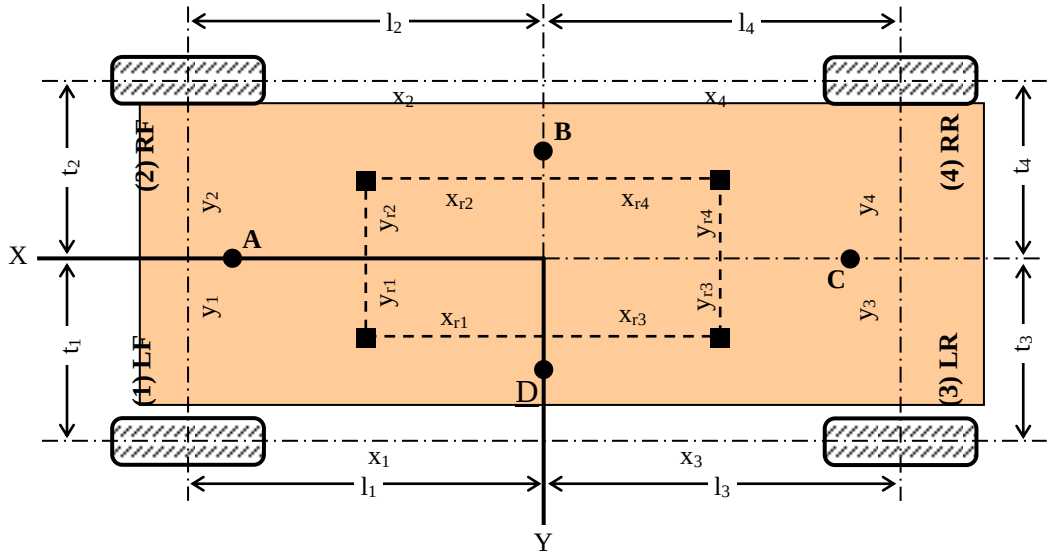
$$AAx + BBy + CCz + DD = 0$$

To determine coefficients AA, BB, CC and DD we use the co-ordinates of any three points on the plane and solve for the determinant shown below:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

We can use any three of the four wheel motions. For the analysis we will use wheel motions (1) and (2) and, to be able to incorporate information from all four corners, a mid-point, C, located between wheel motions (3) and (4) will be our third point used to define the plane.

To determine the motion of the chassis (squat, pitch and roll), we use the vertical change at the origin for squat. To determine the pitch, we use the angle with the horizontal of a line from the origin forward. To determine the roll, we use the angle with the horizontal of a line from the origin sideways.



Location of wheel, ride and mid-points

The equation of the plane containing the four wheel motions will help us determine the elevation (co-ordinate z) of any point of known position (x,y) in that plane. Using the equation of this plane and the co-ordinates of ride points, we can calculate the change in elevation of these ride points. We calculate the chassis motion components squat, pitch and roll using the co-ordinates and elevations of midpoints A, B, C and D.

The co-ordinates of midpoints A, B, C and D are calculated as weighted averages between spring points.

Mid-Point Co-ordinates

	A	**B**	**C**	**D**
x	$\frac{y_1x_2 + y_2x_1}{y_1 + y_2}$	0	$-\frac{y_3x_4 + y_4x_3}{y_3 + y_4}$	0
y	0	$-\frac{x_2y_4 + x_4y_2}{x_2 + x_4}$	0	$\frac{x_1y_3 + x_3y_1}{x_1 + x_3}$
z	$\frac{y_1V_2 + y_2V_1}{y_1 + y_2}$	$\frac{x_2V_4 + x_4V_2}{x_2 + x_4}$	$\frac{y_3V_4 + y_4V_3}{y_3 + y_4}$	$\frac{x_1V_3 + x_3V_1}{x_1 + x_3}$

Note: x_i and y_i are the co-ordinates of tire contact points and V_i are the vertical wheel movements

With this information, we can determine the equation of the plane containing spring points 1 and 2 as well as midpoint C. The governing equation becomes (note the use of – y2 in place of y2 since the Y axis is along the center)

$$\begin{vmatrix} x - x_1 & y - y_1 & z - V_1 \\ x_2 - x_1 & -y_2 - y_1 & V_2 - V_1 \\ x_C - x_1 & y_C - y_1 & z_C - V_1 \end{vmatrix} = 0$$

Note that this condition incorporates information from all wheel motions because, in the last row, we use the co-ordinates of midpoint C (x_C, y_C and z_C) which have information from wheel motions (3) and (4). Solving this determinant we get an equation in terms of *x*, *y* and *z*. Collecting terms we find the coefficients of these variables to be:

$$AA = y_1 V_2 + y_2 V_1 + y_C (V_1 - V_2) - z_C(y_1 + y_2)$$

$$BB = - x_1V_2 + x_2V_1 - x_C (V_1 - V_2) + z_C(x_1 - x_2)$$

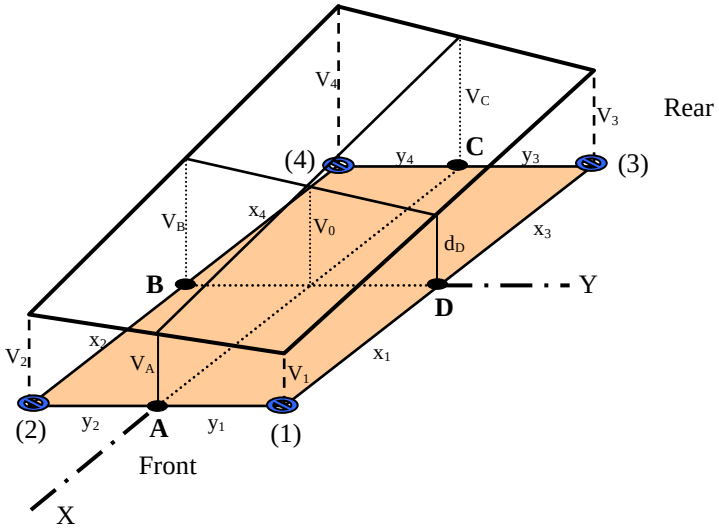
$$CC = - x_1y_2 - x_2y_1 + x_C (y_1 + y_2) + y_C(x_2 - x_1)$$

$$DD = x_C (-y_1V_2 - y_2V_1) + y_C (x_1V_2 - x_2 V_1) + z_C (y_1 x_2 + y_2 x_1)$$

The equation of the plane is $AAx + BBy + CCz + DD = 0$

Chassis Motion Components

With known wheel motions V_1, V_2, V_3 and V_4, determined from an equilibrium analysis, we can calculate the rigid body motions (squat, pitch and roll) of the chassis with reference to the origin and midpoints A and B.



Squat.

To evaluate the overall vertical motion, we will consider the change in vertical co-ordinate of the origin (V_0). To determine this, we plug in the *x* and *y* co-ordinates of the origin into the equation of the plane and find the third co-ordinate (z).

Squat = Change in z-co-ordinate of the origin = V_0

Co-ordinates of the origin:

x = 0, y = 0 z = V_0

Equation of the plane:

AA (0) + BB (0) + CC (V_0) – DD = 0

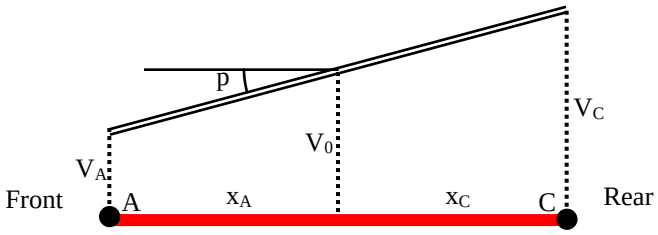
Solving for V_0

$$V_0 = \frac{-DD - AA(0) - BB(0)}{CC} = \frac{+DD}{CC} \rightarrow \text{Squat}$$

Pitch

In general terms, pitch is the vertical angle in the direction of motion of the car, described by the vehicle as its nose goes up or down. To evaluate the pitch, we use the vertical angle between a line from the origin to point A in the front of the car and a horizontal line parallel to the X-axis. For small deformations, this angle, expressed in radians, is calculated as the ratio of the difference in elevation between mid-point A and the origin divided by the x–co-ordinate of point A (x_A).

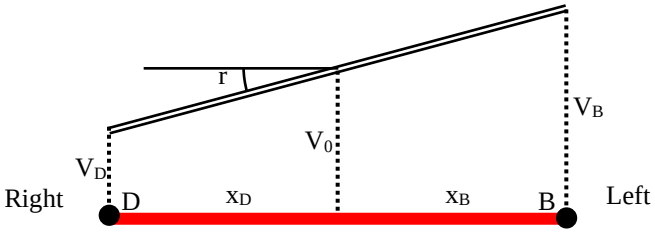
$$p = \frac{V_A - V_0}{x_A}$$



Roll

Roll is the vertical angle, in a direction perpendicular to the motion of the car, described by the vehicle as one side edge goes up and the other side edge goes down when the chassis rotates about the X-axis. To evaluate roll, we use the vertical angle between a line from the origin to point B on the right side of the car and a horizontal line parallel to the Y-axis. For small deformations this angle, expressed in radians, is calculated as the ratio of the difference in elevation between midpoint B and the origin, divided by the y –co-ordinate of point B (y_B).

$$r = \frac{V_B - V_0}{y_B}$$



Ride

The change in ride for an arbitrary point of known position (x_r, y_r) is the vertical co-ordinate (z_r) of that point. To determine the ride change at a point, we plug the *x* and *y* co-ordinates of this point into the equation of the plane to obtain the z-co-ordinate (ride change).

$$\text{ride change} = z_r = -\frac{DD + AAx_r + BBy_r}{CC}$$

While the points used to measure ride (ride points) are not actually in the same plane as the wheel points, they are part of the same rigid body and the change in elevation of these ride points would be the same as the change in elevation of their projection on the plane of the wheel points given by their *x* and *y* co-ordinates.

$$z_{r1} = -\frac{DD + AAx_{r1} + BBy_{r1}}{CC} \qquad z_{r2} = -\frac{DD + AAx_{r2} - BBy_{r2}}{CC}$$

$$z_{r3} = -\frac{DD - AAx_{r3} + BBy_{r3}}{CC} \qquad z_{r4} = -\frac{DD - AAx_{r4} - BBy_{r4}}{CC}$$

Base Case (W_T)

We will now proceed to determine wheel loads and wheel motions for a car at rest, under the action of its own weight. Under these conditions, we will consider only the effect of springs. The eight equations presented to describe equilibrium, wheel load and compatibility of travel can now be written as follows:

[1] $P_1 + P_2 + P_3 + P_4 = W_T$

[2] $P_1 t_1 - P_2 t_2 + P_3 t_3 - P_4 t_4 = W_T y_T$

[3] $P_1 l_1 + P_2 l_2 - P_3 l_3 - P_4 l_4 = W_T x_T$

} Equilibrium

[4] $P_1 = K_1 (V_1 + R_{C1}) + m_1 k_1 D_1 + w_{u1}$

[5] $P_2 = K_2 (V_2 + R_{C2}) + m_2 k_2 D_2 + w_{u2}$

[6] $P_3 = K_3 (V_3 + R_{C3}) + m_3 k_3 D_3 + w_{u3}$

[7] $P_4 = K_4 (V_4 + R_{C4}) + m_4 k_4 D_4 + w_{u4}$

} Wheel Load

[8] $C_1 V_1 + C_2 V_2 + C_3 V_3 + C_4 V_4 = 0$ } Compatibility

where

P_i = Wheel loads

W_T = Total weight

w_{uj} = Unsprung weight (wheel assembly)

V_i = wheel motion

D_i = Spring Preload

R_{Ci} = Change in tire radius

t_i, l_i → Half-track and half-wheel base – Same as x_i, y_i accept for sign

x_T, y_T → CG co-ordinates accept for sign

m_i, k_i → Motion ratio and spring rate

C_i → Compatibility coefficients

$C_1 = (-x_3 + x_4)\, y_2 + (x_2 + x_4)\, y_3 + (x_2 + x_3)\, y_4$

$C_2 = (x_4 - x_3)\, y_1 - (x_1 + x_4)\, y_3 - (x_1 + x_3)\, y_4$

$C_3 = -(x_2 + x_4)\, y_1 - (x_1 + x_4)\, y_2 - (-x_1 + x_2) y_4$

$C_4 = (x_2 + x_3)\, y_1 + (x_1 + x_3)\, y_2 + (x_1 - x_2)\, y_3$

x_i, y_I →Wheel location co-ordinates

System Equations (W_T)

The system of equations used to carry out the analysis is:

[1] $P_1 + P_2 + P_3 + P_4 = W_T$

[2] $P_1 t_1 - P_2 t_2 + P_3 t_3 - P_4 t_4 = W_T y_T$

[3] $P_1 l_1 + P_2 l_2 - P_3 l_3 - P_4 l_4 = W_T x_T$

[4] $P_1 - K_1 V_1 = m_1 k_1 (D_1 + m_1 R_{C1}) + w_{u1}$

[5] $P_2 - K_2 V_2 = m_2 k_2 (D_2 + m_2 R_{C2}) + w_{u1}$

[6] $P_3 - K_3 V_3 = m_3 k_3 (D_3 + m_3 R_{C3}) + w_{u1}$

[7] $P_4 - K_4 V_4 = m_4 k_4 (D_4 + m_4 R_{C4}) + w_{u1}$

[8] $C_1 V_1 + C_2 V_2 + C_3 V_3 + C_4 V_4 = 0$

C_i = Compatibility Coefficients

Matrix Formulation

This system of 8 equations and 8 unknowns, four wheel loads and four wheel motions can be written in matrix form:

$$\mathbf{A}\, X = \mathbf{B}$$

Where

A = Matrix of coefficients for variables P_1, P_2, P_3, P_4, V_1, V_2, V_3, and V_4 in equations [1] through [8]

X = Vector containing variables P_1, P_2, P_3, P_4, V_1, V_2, V_3, and V_4

B = Vector containing constant coefficients on the right hand side of equations [1] through [8]

The system can be written in matrix form, as follows

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ t_1 & -t_2 & t_3 & -t_4 & 0 & 0 & 0 & 0 \\ l_1 & l_2 & -l_3 & -l_4 & 0 & 0 & 0 & 0 \\ E_1 & 0 & 0 & 0 & -K_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 & 0 & -K_2 & 0 & 0 \\ 0 & 0 & E_3 & 0 & 0 & 0 & -K_3 & 0 \\ 0 & 0 & 0 & E_4 & 0 & 0 & 0 & -K_4 \\ 0 & 0 & 0 & 0 & C_1 & C_2 & C_3 & C_4 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{Bmatrix} = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{Bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ t_1 & -t_2 & t_3 & -t_4 & 0 & 0 & 0 & 0 \\ l_1 & l_2 & -l_3 & -l_4 & 0 & 0 & 0 & 0 \\ E_1 & 0 & 0 & 0 & -K_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 & 0 & -K_2 & 0 & 0 \\ 0 & 0 & E_3 & 0 & 0 & 0 & -K_3 & 0 \\ 0 & 0 & 0 & E_4 & 0 & 0 & 0 & -K_4 \\ 0 & 0 & 0 & 0 & C_1 & C_2 & C_3 & C_4 \end{bmatrix}$$

$E_i = 1$ for i = 1 … 4

$K_i = m_i^2 k_i$ for i = 1 … 4

$C_1 = (-x_3 + x_4) y_2 + (x_2 + x_4) y_3 + (x_2 + x_3) y_4$

$C_2 = (x_4 - x_3) y_1 - (x_1 + x_4) y_3 - (x_1 + x_3) y_4$

$C_3 = -(x_2 + x_4) y_1 - (x_1 + x_4) y_2 - (-x_1 + x_2) y_4$

$C_4 = (x_2 + x_3) y_1 + (x_1 + x_3) y_2 + (x_1 - x_2) y_3$

$$X = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{Bmatrix} \quad \text{and} \quad B = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{Bmatrix}$$

and

$B_1 = W_T$

$B_2 = W_T y_T$

$B_3 = W_T x_T$

$B_4 = m_1 k_1 (D_1 + m_1 R_{C1}) + w_{u1}$

$B_5 = m_2 k_2 (D_2 + m_2 R_{C2}) + w_{u2}$

$B_6 = m_3 k_3 (D_3 + m_3 R_{C3}) + w_{u3}$

$B_7 = m_4 k_4 (D_4 + m_4 R_{C4}) + w_{u4}$

$B_8 = 0$

Solution

$$X = \mathbf{A}^{-1}\,\mathbf{B}$$

The inverse of matrix **A**, shown above as $\mathbf{A}^{-1}$ and the matrix multiplication are implemented in an Excel spreadsheet to evaluate the wheel loads P_1, P_2, P_3 and P_4 and the wheel motion V_1, V_2, V_3 and V_4. With known wheel motion and established geometry (track and wheelbase), we can calculate the co-ordinates of midpoints between corners and the coefficients of the equation of the plane containing these points. With the equation of the plane and known co-ordinates of ride points, we can then calculate ride changes. Chassis motion components (squat, pitch and roll) are calculated using midpoint co-ordinates and deflections.

Initial Set up

When a car is first assembled, it will not exhibit the desired performance characteristics of bias, wedge and ride. To meet target values of these parameters, there are a number of procedures to be followed. First, while the car is placed on scale pads to measure wheel loads, weights are added or repositioned at different locations until the right values for left and rear bias are achieved. At this stage, the wedge will probably have not achieved its target value, but this will be resolved later by adjusting spring preloads. We can then calculate spring deformation from $d_L = m_i V_i$

The important thing to remember is that with the present geometry (track, wheel base, spring rates, motion ratios) and configuration of wheel loads, left and rear bias, we have established the location of the centre of gravity (CG) of the car which will be used in future calculations. The co-ordinates of the CG are now calculated using known values of wheel loads, track and wheel base. Knowing the location of the CG for the car as a whole and the location of the CG for the unsprung weight, we can now determine the location of the CG for the sprung weight.

The base case model developed here can be used to simulate the adjustment of spring preloads to achieve desired values for the wedge.

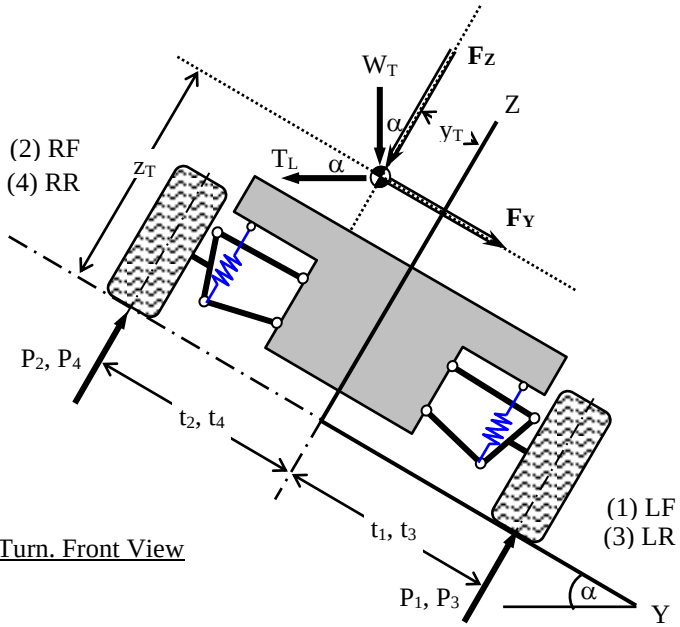
Base Case (W_T, T_L, T_R, Banking)

Introduction

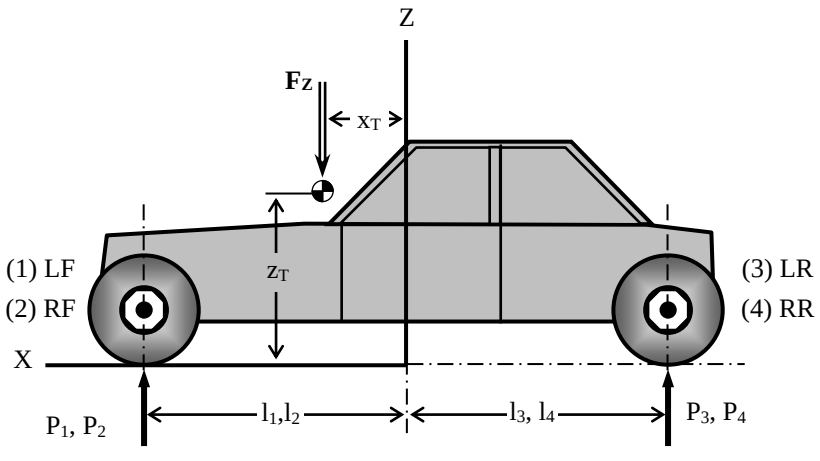
In the previous case, we considered only the vertical force W_T when calculating wheel loads and spring deformations. We will now add inertial forces induced by turning left or right, T_L and T_R, and incorporate the effects of road banking through an angle α. Turning forces are measured in *g's*, that is, in units of total weight, W_T. They tend to make the car overturn in a sense opposite to the sense of the turn, that is, a car turning left tends to overturn to the right. Banking of the road affects both vertical and horizontal forces on the wheel loads because it induces a horizontal component to vertical forces like W_T and a vertical component to the horizontal turning forces, T_L or T_R. At this stage, we still assume that only the springs are acting, ignoring the effects of other suspension components.

Left Turn and Bank

Banking of the road when turning helps maintain stability and affects the distribution of load to each wheel. For this case, we will write equations of equilibrium, spring behaviour and compatibility of deformations incorporating the effect of left or right turn, when the road banks through an angle α.



Left Turn. Front View



Left Turn. Side View

As shown, the system rotates through the bank angle, α, maintaining the orientation of the reference co-ordinate system. Now the vertical load W_T has components in the direction of the Z−axis and in the direction of the Y−axis. The turning force T_L also has components in the direction of both reference axes. Wheel loads P_i remain perpendicular to the road surface and in the direction of the Z-axis. To simplify the writing of the equilibrium equations, we will combine the components of W_T and T_L in the direction of the Y and Z axes as shown below.

$$F_Y = W_T \text{ Sin } \alpha - T_L \text{ Cos } \alpha$$

$$F_Z = W_T \text{ Cos } \alpha + T_L \text{ Sin } \alpha$$

We will now proceed to write the equations of equilibrium, suspension component behaviour and compatibility of spring deformations.

Equilibrium

$\Sigma F_Z = 0$ $\quad P_1 + P_2 + P_3 + P_4 = \mathbf{F_Z}$

[1] $\quad P_1 + P_2 + P_3 + P_4 = W_T \text{ Cos } \alpha + T_L \text{ Sin } \alpha$

$\Sigma M_X = 0$ $\quad P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = \mathbf{F_Y}\, z_T - \mathbf{F_Z}\, y_T$

[2] $\quad P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = (W_T \text{ Sin } \alpha - T_L \text{ Cos } \alpha)\, z_T - (W_T \text{ Cos } \alpha + T_L \text{ Sin } \alpha)\, y_T$

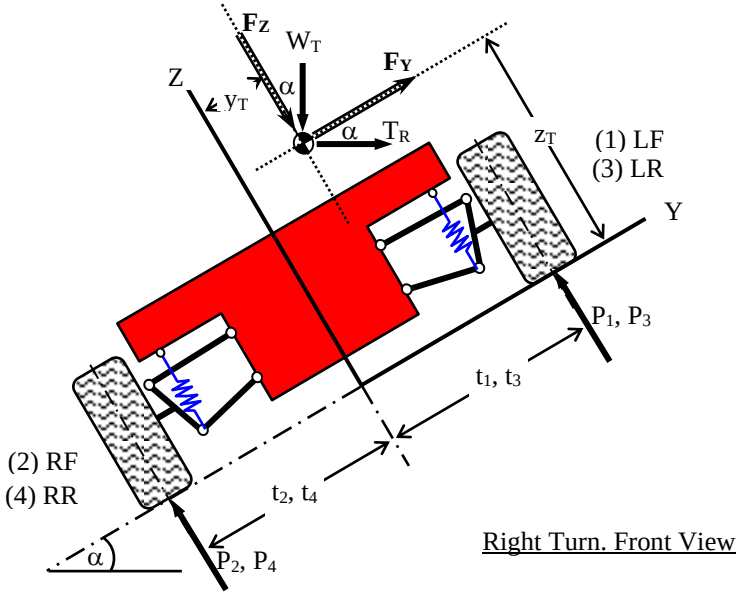
$\Sigma M_Y = 0$ $\quad P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = \mathbf{F_Z}\, x_T$

[3] $\quad P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = (W_T \text{ Cos } \alpha + T_L \text{ Sin } \alpha)\, x_T$

where

P_i	Wheel loads
W_T	Total weight
$\mathbf{F_Y}$	Resultant force in the Y−direction
$\mathbf{F_Z}$	Resultant force in the Z−direction
T_L	Left turn inertial force
t_i	Half track
l_i	Half wheelbase
x_T, y_T, z_T	CG co-ordinates
α	Bank angle

Right Turn and Bank



Right Turn. Front View

$F_Y = -W_T \sin\alpha + T_R \cos\alpha$

$F_Z = W_T \cos\alpha + T_R \sin\alpha$

Equilibrium

$\Sigma F_Z = 0$ $\quad P_1 + P_2 + P_3 + P_4 = \mathbf{F_Z}$

[1] $\quad P_1 + P_2 + P_3 + P_4 = W_T \cos\alpha + T_R \sin\alpha$

$\Sigma M_X = 0$ $\quad P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = \mathbf{F_Y}\, z_T + \mathbf{F_Z}\, y_T$

[2] $\quad P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = -(W_T \sin\alpha - T_R \cos\alpha)\, z_T + (W_T \cos\alpha + T_R \sin\alpha)\, y_T$

$\Sigma M_Y = 0$ $\quad P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = \mathbf{F_Z}\, x_T$

[3] $\quad P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = (W_T \cos\alpha + T_R \sin\alpha)\, x_T$

Component Behaviour and Compatibility

For this case, we use the same relationships developed before and reported as Equations [4] through [8]

Simplified Equations (W_T, Bank)

Since the inertial forces T_L and T_R can not occur simultaneously, we can use the same equilibrium equations for both cases if we write them in the following form:

[1] $P_1 + P_2 + P_3 + P_4 = W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha$

[2] $P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = [-R] \{(W_T \text{ Sin } \alpha - (T_L + T_R) \text{ Cos } \alpha) z_T - (W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha) y_T\}$

[3] $P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = (W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha) x_T$

[4] $P_1 - K_1V_1 = m_1k_1 (D_1 + m_1R_{C1}) + w_{u1}$

[5] $P_2 - K_2V_2 = m_2k_2 (D_2 + m_2R_{C2}) + w_{u1}$

[6] $P_3 - K_3V_3 = m_3k_3 (D_3 + m_3R_{C3}) + w_{u1}$

[7] $P_4 - K_4V_4 = m_4k_4 (D_4 + m_4R_{C4}) + w_{u1}$

[8] $C_1 V_1 + C_2 V_2 + C_3 V_3 + C_4 V_4 = 0$

where

[–R] = Negative sign for Right turn

C_i = Compatibility coefficients

x_i, y_i = Wheel location co-ordinates

Matrix Formulation

As shown previously, this system of 8 equations and 8 unknowns, four wheel loads and four spring deformations, can be written in matrix form, **A** X = **B**, with

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ t_1 & -t_2 & t_3 & -t_4 & 0 & 0 & 0 & 0 \\ l_1 & l_2 & -l_3 & -l_4 & 0 & 0 & 0 & 0 \\ E_1 & 0 & 0 & 0 & -K_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 & 0 & -K_2 & 0 & 0 \\ 0 & 0 & E_3 & 0 & 0 & 0 & -K_3 & 0 \\ 0 & 0 & 0 & E_4 & 0 & 0 & 0 & -K_4 \\ 0 & 0 & 0 & 0 & C_1 & C_2 & C_3 & C_4 \end{bmatrix}$$

where

$E_i = 1$ for i = 1 … 4

$K_i = m_i^2 k_i$ for i = 1 … 4

C_i = Compatibility Coefficients

$$B = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{Bmatrix}$$

where

$B_1 = W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha$

$B_2 = [-R] \{(W_T \text{ Sin } \alpha - (T_L + T_R) \text{ Cos } \alpha) z_T - (W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha) y_T\}$

$B_3 = (W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha) x_T$

$B_4 = m_1 k_1 (D_1 + m_1 R_{C1}) + w_{u1}$

$B_5 = m_2 k_2 (D_2 + m_2 R_{C2}) + w_{u2}$

$B_6 = m_3 k_3 (D_3 + m_3 R_{C3}) + w_{u3}$

$B_7 = m_4 k_4 (D_4 + m_4 R_{C4}) + w_{u4}$

$B_8 = 0$

[–R] → Use negative sign for right turn.

Solving this system of equations, $X = \mathbf{A}^{-1} \mathbf{B}$, results in 4 wheel loads and 4 wheel motions. The wheel loads are used to calculate left, rear and wedge bias (B_L, B_R, B_W). The wheel motions are used to determine the equation for the plane of the wheel locations, which, in turn, is used to determine ride changes at the ride points. Wheel motions are also used to find vertical displacements at the spring midpoints, for a given geometry, and then to determine the chassis motion components: squat, pitch and roll.

Base Case (Acceleration and Braking)

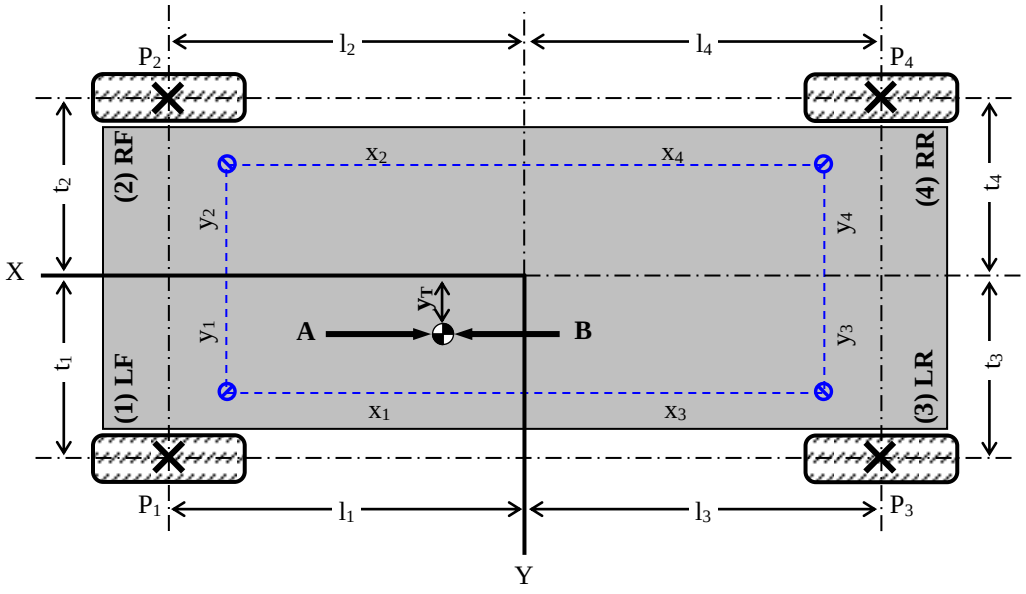
Introduction

In the previous cases, we considered vertical load W_T and lateral inertial forces, T_L and T_R, induced by turning when calculating wheel loads and wheel motions. We now consider the action of longitudinal inertial forces induced by acceleration and braking in the direction of motion.

Acceleration and Braking

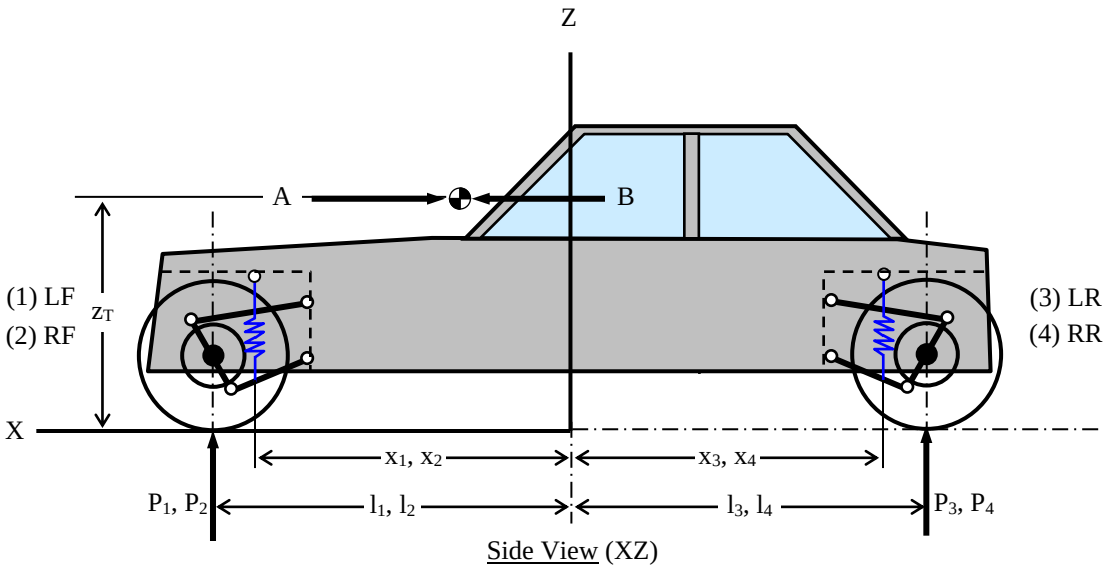
Like turning forces, acceleration and braking forces are measured in *g's*, that is, in units of total weight, W_T. Acceleration tends to make the nose of the car go up and the rear of the car go down, reducing the wheel loads at the front and increasing the wheel loads at the rear. Braking has the opposite effect. Both acceleration and braking forces are horizontal, parallel to the X–axis and are assumed to be applied at the CG. As for the previous case, we assume that only the springs are present, ignoring the effects of the other suspension components at this stage.

The figures below show the location of acceleration, braking and wheel loads in plan and side views. This allows us to see the location of these loads when writing the equations required for the analysis.



Plan View (XY)

Acceleration, Braking and Wheel Loads



Side View (XZ)

Acceleration, Braking and Wheel Loads

Equilibrium

$\Sigma F_Z = 0$

[1] $P_1 + P_2 + P_3 + P_4 = 0$

$\Sigma M_X = 0$

[2] $P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = 0$

$\Sigma M_Y = 0$

[3] $P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = (B - A)\, z_T$

Component Behaviour and Compatibility

For this case, we use the same relationships developed before and reported as Equations [4] through [8]

Simplified Equations (A,B)

[1] $P_1 + P_2 + P_3 + P_4 = 0$

[2] $P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = 0$

[3] $P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = (B - A)\, z_T$

[4] $P_1 - K_1V_1 = m_1k_1\,(D_1 + m_1R_{C1}) + w_{u1}$

[5] $P_2 - K_2V_2 = m_2k_2\,(D_2 + m_2R_{C2}) + w_{u1}$

[6] $P_3 - K_3V_3 = m_3k_3\,(D_3 + m_3R_{C3}) + w_{u1}$

[7] $P_4 - K_4V_4 = m_4k_4\,(D_4 + m_4R_{C4}) + w_{u1}$

[8] $C_1\, d_1 + C_2\, d_2 + C_3\, d_3 + C_4\, d_4 = 0$

Matrix Formulation

As shown previously, this system of 8 equations and 8 unknowns, four wheel loads and four spring deformations, can be written in matrix form, **A** X = **B**, with

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ t_1 & -t_2 & t_3 & -t_4 & 0 & 0 & 0 & 0 \\ l_1 & l_2 & -l_3 & -l_4 & 0 & 0 & 0 & 0 \\ E_1 & 0 & 0 & 0 & -K_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 & 0 & -K_2 & 0 & 0 \\ 0 & 0 & E_3 & 0 & 0 & 0 & -K_3 & 0 \\ 0 & 0 & 0 & E_4 & 0 & 0 & 0 & -K_4 \\ 0 & 0 & 0 & 0 & C_1 & C_2 & C_3 & C_4 \end{bmatrix}$$

where

$E_i = 1$ for i = 1 … 4

$K_i = m_i^2 k_i$ for i = 1 … 4

C_i = Compatibility coefficients

$$X = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{Bmatrix} \quad \text{and} \quad B = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{Bmatrix}$$

where

$B_1 = 0$

$B_2 = 0$

$B_3 = (B-A)\, z_T$

$B_4 = m_1 k_1 (D_1 + m_1 R_{C1}) + w_{u1}$

$B_5 = m_2 k_2 (D_2 + m_2 R_{C2}) + w_{u2}$

$B_6 = m_3 k_3 (D_3 + m_3 R_{C3}) + w_{u3}$

$B_7 = m_4 k_4 (D_4 + m_4 R_{C4}) + w_{u4}$

$B_8 = 0$

Solving this system of equations, $X = \mathbf{A}^{-1}\,\mathbf{B}$, results in 4 wheel loads and 4 wheel motions. The wheel loads determined here are added to those calculated for the previous case and the total is used to calculate left, rear and wedge bias (B_L, B_R, B_W). The wheel motions calculated here are also added to those determined for the previous case and used to determine the equation for the plane of the wheel locations, which, in turn, is used to determine ride changes at the ride points. Total wheel motions, from lateral and longitudinal inertial forces are used to find vertical displacements at the spring midpoints, for a given geometry, and then to determine the chassis motion components: squat, pitch and roll.

Base Case (Other Loads and Suspension Components)

Introduction

Thus far, we have considered the effects of the vertical force W_T, turning forces T_L and T_R, road banking through an angle α, acceleration force A and braking force B, when calculating wheel loads and spring deformations. We will now consider the effect of other loads such as:

- Ballast
- Driver
- Fuel
- Aerodynamic

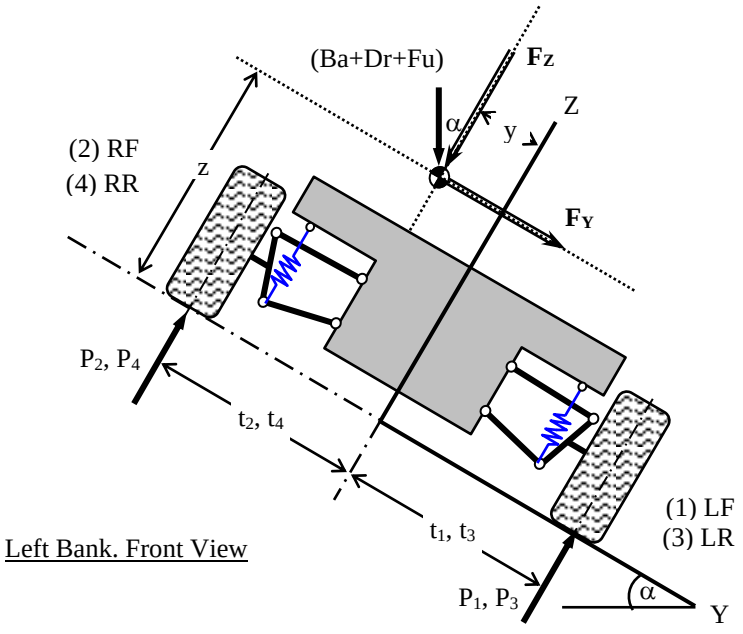
Ballast is added to the car when we need to adjust the location of the CG. Driver weight adds to the mass of the car and often changes during a race. Fuel weight also changes during the race and so does its CG. Aerodynamic forces change with velocity and other geometric parameters.

Load		CG		
		X	**Y**	**Z**
Ballast	Ba	x_B	y_B	z_B
Driver	Dr	x_D	y_D	z_D
Fuel	Fu	x_F	y_F	z_F
Aero	Ae	x_A	y_A	z_A

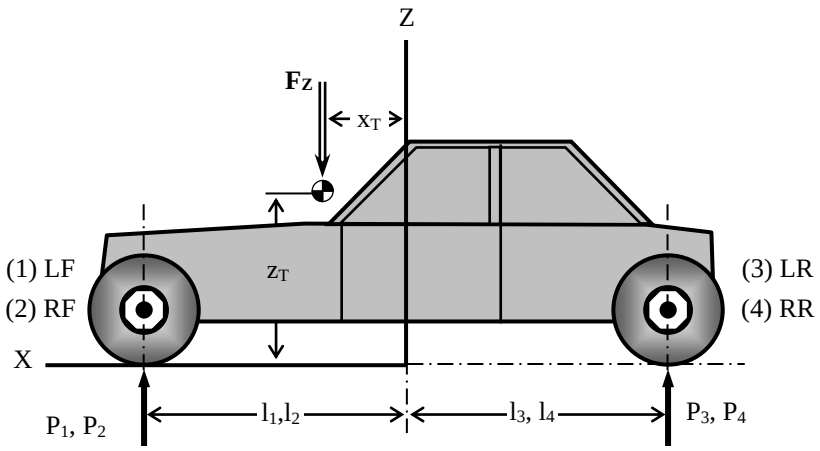
As for the previous cases, we assume that only the springs act as suspension components, ignoring the effects of the other elements. All these are vertical loads and can be treated in a similar way as when we analyzed the effect of the weight, W_T. For this reason, we need to know their magnitude and location co-ordinates for their own CG.

Left Bank

Banking of the road introduces components of these forces in the Y and Z directions. Aerodynamic loads act normal to the track surface.



Left Bank. Front View



Side View (XZ)

$F_Y = (Ba + Dr + Fu) \sin \alpha$

$F_Z = (Ba + Dr + Fu) \cos \alpha + Ae$

Equilibrium

$\Sigma F_Z = 0$ $\quad P_1 + P_2 + P_3 + P_4 = \mathbf{F_Z}$

[1] $\quad P_1 + P_2 + P_3 + P_4 = (Ba + Dr + Fu) \cos \alpha + Ae$

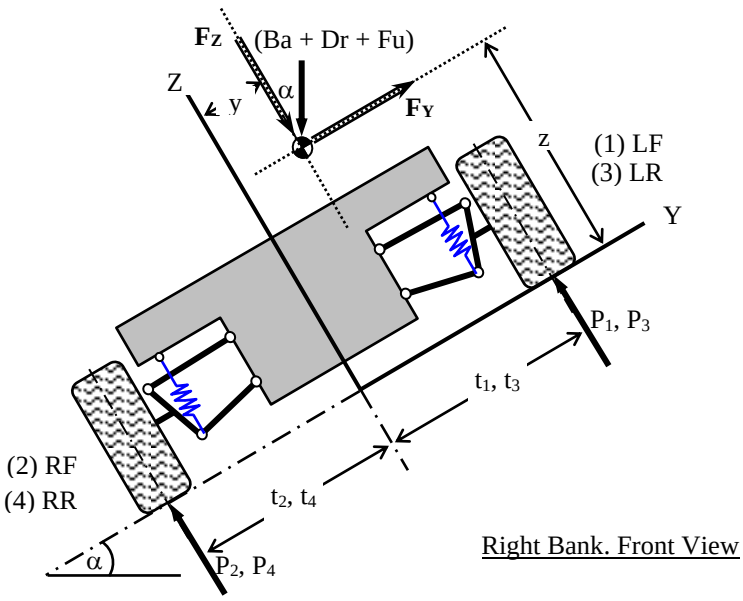
$\Sigma M_X = 0$ $\quad P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = \mathbf{F_Y}\, z - \mathbf{F_Z}\, y$

[2] $\quad P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 =$
$(Ba\, z_B + Dr\, z_D + Fu\, z_F) \sin \alpha - (Ba\, y_B + Dr\, y_D + Fu\, y_F) \cos \alpha + Ae\, y_A$

$\Sigma M_Y = 0$ $\quad P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = \mathbf{F_Z}\, x_T$

[3] $\quad P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = (Ba\, x_B + Dr\, x_D + Fu\, x_F) \cos \alpha + Ae\, x_A$

Right Bank



Right Bank. Front View

$F_Y = -(Ba + Dr + Fu) \sin \alpha$

$F_Z = (Ba + Dr + Fu) \cos \alpha + Ae$

Equilibrium

$\Sigma F_Z = 0$ $\quad P_1 + P_2 + P_3 + P_4 = \mathbf{F_Z}$

[1] $\quad P_1 + P_2 + P_3 + P_4 = (Ba + Dr + Fu) \cos \alpha + Ae$

$\Sigma M_X = 0$ $\quad P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = \mathbf{F_Y}\, z_T + \mathbf{F_Z}\, y_T$

[2] $\quad P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = -(Ba\, z_B + Dr\, z_D + Fu\, z_F) \sin \alpha + (Ba\, y_B + Dr\, y_D + Fu\, y_F) \cos \alpha + Ae\, y_A$

$\Sigma M_Y = 0$ $\quad P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = \mathbf{F_Z}\, x_T$

[3] $\quad P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = (Ba\, x_B + Dr\, x_D + Fu\, x_F) \cos \alpha + Ae\, x_A$

Component Behaviour and Compatibility

Use the same relationships developed before.

Simplified Equations (Other Loads, Banking)

Considering both left and right banking

[1] $\quad P_1 + P_2 + P_3 + P_4 = (Ba + Dr + Fu) \cos \alpha + Ae$

[2] $\quad P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 =$
$[-R]\{(Ba\, z_B + Dr\, z_D + Fu\, z_F) \sin \alpha - (Ba\, y_B + Dr\, y_D + Fu\, y_F) \cos \alpha + Ae\, y_A\}$

[3] $\quad P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = (Ba\, x_B + Dr\, x_D + Fu\, x_F) \cos \alpha + Ae\, x_A$

[4] $P_1 - K_1V_1 = m_1k_1 (D_1 + m_1R_{C1}) + w_{u1}$

[5] $P_2 - K_2V_2 = m_2k_2 (D_2 + m_2R_{C2}) + w_{u1}$

[6] $P_3 - K_3V_3 = m_3k_3 (D_3 + m_3R_{C3}) + w_{u1}$

[7] $P_4 - K_4V_4 = m_4k_4 (D_4 + m_4R_{C4}) + w_{u1}$

[8] $C_1 V_1 + C_2 V_2 + C_3 V_3 + C_4 V_4 = 0$

where

[–R] = Negative sign for Right turn

C_i = Compatibility coefficients

Matrix Formulation

As shown previously, this system of 8 equations and 8 unknowns, four wheel loads and four spring deformations, can be written in matrix form, **A** X = **B**, with

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ t_1 & -t_2 & t_3 & -t_4 & 0 & 0 & 0 & 0 \\ l_1 & l_2 & -l_3 & -l_4 & 0 & 0 & 0 & 0 \\ E_1 & 0 & 0 & 0 & -K_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 & 0 & -K_2 & 0 & 0 \\ 0 & 0 & E_3 & 0 & 0 & 0 & -K_3 & 0 \\ 0 & 0 & 0 & E_4 & 0 & 0 & 0 & -K_4 \\ 0 & 0 & 0 & 0 & C_1 & C_2 & C_3 & C_4 \end{bmatrix}$$

where

$E_i = 1$ for i = 1 … 4

$K_i = m_i^2 k_i$ for i = 1 … 4

C_i = Compatibility coefficients

$$X = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{Bmatrix} \quad \text{and} \quad B = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{Bmatrix}$$

where

$B_1 = (Ba + Dr + Fu) \cos \alpha + Ae$

$B_2 = [-R]\{(Ba\ z_B + Dr\ z_D + Fu\ z_F) \sin \alpha - (Ba\ y_B + Dr\ y_D + Fu\ y_F) \cos \alpha + Ae\ y_A\}$

$B_3 = (Ba\ x_B + Dr\ x_D + Fu\ x_F) \cos \alpha + Ae\ x_A$

$B_4 = m_1 k_1 (D_1 + m_1 R_{C1}) + w_{u1}$

$B_5 = m_2 k_2 (D_2 + m_2 R_{C2}) + w_{u2}$

$B_6 = m_3 k_3 (D_3 + m_3 R_{C3}) + w_{u3}$

$B_7 = m_4 k_4 (D_4 + m_4 R_{C4}) + w_{u4}$

$B_8 = 0$

[–R] → Use negative sign for a right turn

Solving this system of equations, $X = \mathbf{A}^{-1}\ \mathbf{B}$, results in 4 wheel loads and 4 wheel motions. The wheel loads are used to calculate left, rear and wedge bias (B_L, B_R, B_W). The wheel motions are used to determine the equation for the plane of the wheel locations, which, in turn, is used to determine ride changes at the ride points. Wheel motions are also used to find vertical displacements at the wheel midpoints, for a given geometry, and then to determine the chassis motion components: squat, pitch and roll.

Other Suspension Components

Equations [4] through [7] previously derived for a suspension system consisting of springs alone can now be expanded to include roll bars and shocks. A similar procedure as that used for springs can be developed if roll bars and dampers are present, with due consideration to their geometric and material behaviour characteristics. A general formulation incorporating springs, roll bars and dampers is given below:

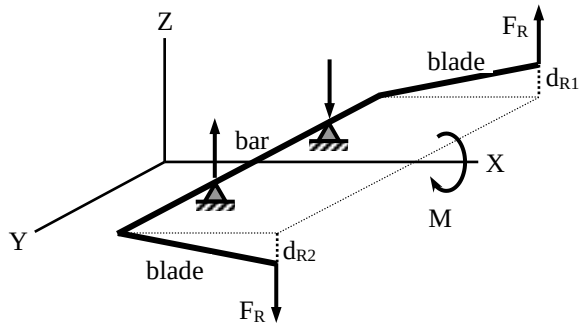
$P = m\ F + m_R\ F_R + m_S\ F_S + w_u$

where

F = Spring force	m = Spring motion ratio
F_R = Roll bar force	m_R = Roll bar motion ratio
F_S = Damper force	m_S = Damper motion ratio

Roll Bar

The roll bar is a mechanism used to reduce lateral rotation (roll) of the chassis under the action of loads induced by a turn or other asymmetrical loads. It is a simple mechanism but its static analysis is definitely not trivial. The roll bar is engaged whenever there is unequal movement of the blades, whatever the source.



The longitudinal, roll-inducing, torque (M) might be the result of a turn or of an eccentric vertical load, such as the weight of the chassis when the y_S co-ordinate of its centre of gravity is not zero. The reaction force at the tip of the blade, F_R, is calculated using the roll bar rate (k_R) and the combined deformation at both ends (d_{R1}, d_{R2}).

$$F_R = k_R (d_{R1} - d_{R2}) = k_R d_R$$

The roll bar acts as a pair of special springs with a common spring rate (k_R) that, in order to be effective, must act in the opposite sense. That is, when one goes up, the other must go down. When a symmetric car pitches, both ends of the blade travel in the same sense, up or down, and their difference $d_{R1} - d_{R2} = 0$, so that the reactive force $F_R = 0$. There is also a compatibility restriction on the amount of 'travel' at the end of each blade and these deformations must comply with the overall behaviour of the rigid chassis and that of the other flexible suspension components.

Other asymmetric deformations, such as preload (D_R) and a radius change (R_C), imposed on the system may induce a torque and engage the roll bar. The roll bar preload (D_R) is applied directly to the blade but the radius change (R_C) occurs at the wheel and must be 'converted' to blade-tip deformation using the motion ratio (m_R). The general formulation for the roll bar is given by

$$F_R = k_R(d_R + D_R + m_R R_C)$$

And the effect of this force on the wheel load is

$$m_R F_R = m_R k_R(d_R + D_R + m_R R_C)$$

Shocks

Shocks, or dampers, are visco-elastic suspension components and their behaviour depends on the applied rate of deformation. While the force on an elastic component, like a spring, depends on the applied deformation, the force on a visco-elastic component, like a damper, depends on the *velocity* of the applied deformation. The calculation of the velocity of the chassis in the direction of the shock is out of the scope of the analysis at this stage and damper forces will be assumed to be known. The total force on the damper may include a preload.

$$F_S = f_S + f_{SP}$$

The effect of this force on the wheel load is

$$m_S F_S = m_S (f_S + f_{SP})$$

A General Formulation

The general equation for suspension component behaviour becomes:

$$P = mk (d + D + mR_C) + m_R k_R (d_R + D_R + m_R R_C) + m_S (f_S + f_{SP}) + w_u$$

The solution of the system of 8 equations and 8 unknowns, when written in matrix form, requires that the equations be written with all unknown variables on the left and all known coefficients on the right hand side of the equal sign. We also replace the spring deformation d with the wheel motion variable V, which is one of the 9 unknowns, and we use K in place of m^2k. Rearranging terms and collecting similar coefficients:

$$P = mkd + mk(D + mR_C) + m_R k_R d_R + m_R k_R(D_R + m_R R_C) + m_S (f_S + f_{SP}) + w_u$$
$$P = mkmV + mk(D + mR_C) + m_R k_R d_R + m_R k_R(D_R + m_R R_C) + m_S (f_S + f_{SP}) + w_u$$
$$P = KV + mk(D + mR_C) + m_R k_R d_R + m_R k_R(D_R + m_R R_C) + m_S (f_S + f_{SP}) + w_u$$

$$P - KV = mk(D + mR_C) + m_R k_R d_R + m_R k_R(D_R + m_R R_C) + m_S (f_S + f_{SP}) + w_u$$
$$P - KV - m_R k_R d_R = mk(D + mR_C) + m_R k_R(D_R + m_R R_C) + m_S (f_S + f_{SP}) + w_u$$

We now have all of the unknowns on the left side and all known values on the right side of the equation.
This last equation, with appropriate subscripts, can now be written for each corner.

[4] $P_1 - K_1V_1 - m_{R1}k_{R1}d_{R1} = m_1k_1 (D_1 + m_1R_{C1}) + m_{R1}k_{R1} (D_{R1} + m_{R1}R_{C1}) + m_{S1}(f_{S1} + f_{SP1}) + w_{u1}$

[5] $P_2 - K_2V_2 - m_{R2}k_{R2}d_{R2} = m_2k_2 (D_2 + m_2R_{C2}) + m_{R2}k_{R2} (D_{R2} + m_{R2}R_{C2}) + m_{S2}(f_{S2} + f_{SP2}) + w_{u2}$

[6] $P_3 - K_3V_3 - m_{R3}k_{R3}d_{R3} = m_3k_3 (D_3 + m_3R_{C3}) + m_{R3}k_{R3} (D_{R3} + m_{R3}R_{C3}) + m_{S3}(f_{S3} + f_{SP3}) + w_{u3}$

[7] $P_4 - K_4V_4 - m_{R4}k_{R4}d_{R4} = m_4k_4 (D_4 + m_4R_{C4}) + m_{R4}k_{R4} (D_{R4} + m_{R4}R_{C4}) + m_{S4}(f_{S4} + f_{SP4}) + w_{u4}$

Roll bar displacements d_{R1}, d_{R2}, d_{R3}, d_{R4} are not independent. First, at the front end, d_{R1} and d_{R2} are combined to determine the total roll bar distortion $d_{RF} = d_{R1} - d_{R2}$. At the rear end, d_{R3} and d_{R4} are combined to determine the total roll bar distortion $d_{RR} = d_{R3} - d_{R4}$. Furthermore, since the chassis is a rigid body the amount of roll at the front must be equal to that at the rear and $d_{RF} = d_{RR} = d_R$. Also, each roll bar has its own stiffness, so we need only two roll bar rates, namely k_{RF} for the front and k_{RR} fore the rear suspension. Equations [4] through [7] are now rewritten:

[4] $P_1 - K_1V_1 - m_{R1}k_{RF}d_R = m_1k_1 (D_1 + m_1R_{C1}) + m_{R1}k_{RF} (D_{R1} + m_{R1}R_{C1}) + m_{S1}(f_{S1} + f_{SP1}) + w_{u1}$

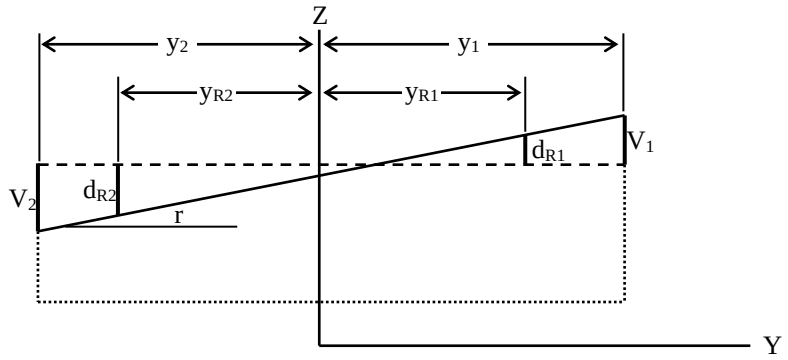
[5] $P_2 - K_2V_2 + m_{R2}k_{RF}d_R = m_2k_2 (D_2 + m_2R_{C2}) + m_{R2}k_{RF} (D_{R2} + m_{R2}R_{C2}) + m_{S2}(f_{S2} + f_{SP2}) + w_{u2}$

[6] $P_3 - K_3V_3 - m_{R3}k_{RR}d_R = m_3k_3 (D_3 + m_3R_{C3}) + m_{R3}k_{RR} (D_{R3} + m_{R3}R_{C3}) + m_{S3}(f_{S3} + f_{SP3}) + w_{u3}$

[7] $P_4 - K_4V_4 + m_{R4}k_{RR}d_R = m_4k_4 (D_4 + m_4R_{C4}) + m_{R4}k_{RR} (D_{R4} + m_{R4}R_{C4}) + m_{S4}(f_{S4} + f_{SP4}) + w_{u4}$

The four rates k_{R1}, k_{R2}, k_{R3}, k_{R4}, are now represented by two: k_{RF} and k_{RR}. The four displacements d_{R1}, d_{R2}, d_{R3}, d_{R4}, are now represented by one: d_R. Notice the sign change for the third term in equations [5] and [7]. It reflects the fact that the roll bar is engaged only when the tip of the two blades move in the opposite sense.

We have now one more variable, d_R, and we need one more equation. This extra equation is to be derived from a condition of compatibility between roll bar deflections and wheel motion.



Roll $$r = \frac{V_1 - V_2}{y_1 + y_2} = \frac{d_{R1} - d_{R2}}{y_{R1} + y_{R2}} = \frac{d_{RF}}{y_{R1} + y_{R2}}$$

[9] $V_1 - V_2 - G\, d_R = 0$

where

$$G = \frac{y_1 + y_2}{y_{R1} + y_{R2}}$$

Equations [4] to [7] now include an extra variable (d_R) and we need to add Equation [9] to our model.

Base Case (All Loads, Banking)

Introduction

Thus far, we have considered separately the effects of the vertical force W_T, turning forces T_L and T_R, road banking through an angle α, acceleration force A, braking force B and other loads such as Ballast, Driver, Fuel and Aerodynamic. We now will bring all these loads together and write the equations describing the system. At this stage all loads are present, the magnitude and location of the CGs for all loads are known, the geometry (track and wheel base) is set and all types of suspension components are included in the model.

Model Equations

[1] $P_1 + P_2 + P_3 + P_4 = W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha + (Ba + Dr + Fu) \text{ Cos } \alpha + Ae$

[2] $P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = [-R]\{(W_T \text{ Sin } \alpha - (T_L + T_R) \text{ Cos } \alpha)\, z_T - (W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha) y_T$
$+(Ba\, z_B + Dr\, z_D + Fu\, z_F + Ae\, z_A) \text{ Sin } \alpha - (Ba\, y_B + Dr\, y_D + Fu\, y_F) \text{ Cos } \alpha + Ae\, y_A\}$

[3] $P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = (W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha)\, x_T$
$+ (B-A)\, z_T + (Ba\, x_B + Dr\, x_D + Fu\, x_F) \text{ Cos } \alpha + Ae\, x_A$

[4] $P_1 - K_1V_1 - m_{R1}k_{RF}d_R = m_1k_1\,(D_1 + m_1R_{C1}) + m_{R1}k_{RF}\,(D_{R1} + m_{R1}R_{C1}) + m_{S1}(f_{S1} + f_{SP1}) + w_{u1}$

[5] $P_2 - K_2V_2 + m_{R2}k_{RF}d_R = m_2k_2\,(D_2 + m_2R_{C2}) + m_{R2}k_{RF}\,(D_{R2} + m_{R2}R_{C2}) + m_{S2}(f_{S2} + f_{SP2}) + w_{u2}$

[6] $P_3 - K_3V_3 - m_{R3}k_{RR}d_R = m_3k_3\,(D_3 + m_3R_{C3}) + m_{R3}k_{RR}\,(D_{R3} + m_{R3}R_{C3}) + m_{S3}(f_{S3} + f_{SP3}) + w_{u3}$

[7] $P_4 - K_4V_4 + m_{R4}k_{RR}d_R = m_4k_4\,(D_4 + m_4R_{C4}) + m_{R4}k_{RR}\,(D_{R4} + m_{R4}R_{C4}) + m_{S4}(f_{S4} + f_{SP4}) + w_{u4}$

[8] $C_1\, V_1 + C_2\, V_2 + C_3\, V_3 + C_4\, V_4 = 0$

[9] $V_1 - V_2 - G\, d_R = 0$

Matrix Formulation

As shown previously, this system of equations can be written in matrix form, **A** X = **B**, with

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ t_1 & -t_2 & t_3 & -t_4 & 0 & 0 & 0 & 0 & 0 \\ l_1 & l_2 & -l_3 & -l_4 & 0 & 0 & 0 & 0 & 0 \\ E_1 & 0 & 0 & 0 & -K_1 & 0 & 0 & 0 & K_{R1} \\ 0 & E_2 & 0 & 0 & 0 & -K_2 & 0 & 0 & K_{R2} \\ 0 & 0 & E_3 & 0 & 0 & 0 & -K_3 & 0 & K_{R3} \\ 0 & 0 & 0 & E_4 & 0 & 0 & 0 & -K_4 & K_{R4} \\ 0 & 0 & 0 & 0 & C_1 & C_2 & C_3 & C_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -G \end{bmatrix}$$

where

$E_i = 1$ for i = 1 … 4

$K_i = m_i^2\, k_i$ for i = 1 … 4

$K_{R1} = -m_{R1}k_{RF}$ $K_{R2} = m_{R2}k_{RF}$ $K_{R3} = -m_{R3}k_{RR}$ $K_{R4} = m_{R4}k_{RR}$

C_i = Compatibility coefficients

$$G = \frac{y_1 + y_2}{y_{R1} + y_{R2}}$$

$$X = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ d_R \end{Bmatrix} \quad \text{and} \quad B = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \\ B_9 \end{Bmatrix}$$

where

$B_1 = W_T \cos\alpha + (T_L + T_R) \sin\alpha + (Ba + Dr + Fu) \cos\alpha + Ae$

$B_2 = [-R]\{(W_T \sin\alpha - (T_L + T_R) \cos\alpha)\, z_T - (W_T \cos\alpha + (T_L + T_R) \sin\alpha)\, y_T$
$(Ba\, z_B + Dr\, z_D + Fu\, z_F + Ae\, z_A) \sin\alpha - (Ba\, y_B + Dr\, y_D + Fu\, y_F) \cos\alpha + Ae\, y_A\}$

$B_3 = (W_T \cos\alpha + (T_L + T_R) \sin\alpha)\, x_T + (B - A)\, z_T + (Ba\, x_B + Dr\, x_D + Fu\, x_F) \cos\alpha + Ae\, y_A$

$B_4 = m_1 k_1 (D_1 + m_1 R_{C1}) + m_{R1} k_{RF} (D_{R1} + m_{R1} R_{C1}) + m_{S1}(f_{S1} + f_{SP1}) + w_{u1}$

$B_5 = m_2 k_2 (D_2 + m_2 R_{C2}) + m_{R2} k_{RF} (D_{R2} + m_{R2} R_{C2}) + m_{S2}(f_{S2} + f_{SP2}) + w_{u2}$

$B_6 = m_3 k_3 (D_3 + m_3 R_{C3}) + m_{R3} k_{RR} (D_{R3} + m_{R3} R_{C3}) + m_{S3}(f_{S3} + f_{SP3}) + w_{u3}$

$B_7 = m_4 k_4 (D_4 + m_4 R_{C4}) + m_{R4} k_{RR} (D_{R4} + m_{R4} R_{C4}) + m_{S4}(f_{S4} + f_{SP4}) + w_{u4}$

$B_8 = 0$

$B_9 = 0$

[–R] → Use negative sign for right turn

Solving this system of equations, $X = \mathbf{A}^{-1}\,\mathbf{B}$, results in 4 wheel loads, 4 spring deformations and one roll bar distortion.

$P_1, P_2, P_3, P_4, V_1, V_2, V_3, V_4, d_R$

The wheel loads are used to calculate left, rear and wedge bias (B_L, B_R, B_W).

$$B_L = \frac{P_1 + P_3}{W_T} \qquad B_R = \frac{P_3 + P_4}{W_T} \qquad B_w = \frac{P_2 + P_3}{W_T}$$

With known geometry (spring co-ordinates x and y), the wheel motions are used to determine the co-ordinates of midpoints A, B, C and D.

Mid-Point Co-ordinates

	A	**B**	**C**	**D**
x	$\frac{y_1x_2 + y_2x_1}{y_1 + y_2}$	0	$-\frac{y_3x_4 + y_4x_3}{y_3 + y_4}$	0
y	0	$-\frac{x_2y_4 + x_4y_2}{x_2 + x_4}$	0	$\frac{x_1y_3 + x_3y_1}{x_1 + x_3}$
z	$\frac{y_1V_2 + y_2V_1}{y_1 + y_2}$	$\frac{x_2V_4 + x_4V_2}{x_2 + x_4}$	$\frac{y_3V_4 + y_4V_3}{y_3 + y_4}$	$\frac{x_1V_3 + x_3V_1}{x_1 + x_3}$

Note: x_i and y_i are the co-ordinates of tire contact points and V_i are the vertical wheel movements

These midpoint co-ordinates will be used to find the equation for the plane of the wheel locations.

$$AAx + BBy + CCz + DD = 0$$

Where

$$AA = y_1 V_2 + y_2 V_1 + y_C (V_1 - V_2) - z_C(y_1 + y_2)$$

$$BB = - x_1V_2 + x_2V_1 - x_C (V_1 - V_2) + z_C(x_1 - x_2)$$

$$CC = - x_1y_2 - x_2y_1 + x_C (y_1 + y_2) + y_C(x_2 - x_1)$$

$$DD = x_C (-y_1d_2 - y_2d_1) + y_C (x_1d_2 - x_2 d_1) + z_C (y_1 x_2 + y_2 x_1)$$

The equation of the plane is used to determine ride changes at the ride points. We substitute the co-ordinates x_r and y_r for each ride point into the equation of the plane to find its change in elevation, z_r.

$$z_r = \frac{-DD - AAx_r - BBy_r}{CC}$$

Wheel Motions are also used to find vertical displacements at the wheel midpoints, d_A, d_B, d_C, d_D, for a given geometry, and then to determine the chassis motion components: squat, pitch and roll. First, we find the change in ride for the origin (x = 0 and y = 0).

$$d_0 = z_0 = \frac{-DD - AA(0) - BB(0)}{CC} = -\frac{DD}{CC}$$

Next we use the change in elevation for midpoints A and B, d_A and d_B, to find the motion components

$$\textit{Squat} \rightarrow V_0 \qquad \textit{Pitch} \rightarrow p = \frac{d_A - d_0}{x_A} \qquad \textit{Roll} \rightarrow r = \frac{d_B - d_0}{y_B}$$

Anti Forces

Review

Under the action of vertical (W_T, Ba, Dr, Fu and Ae), lateral (T_L or T_R) or longitudinal (A and B) forces, the chassis transmits loads to the contact patch under the wheels through the suspension components such as: springs, roll bars, dampers, track bars and other links.

Springs and roll bars are flexible and elastic components that must deform before they can transmit the loads from the chassis. The amount of deformation depends on their stiffness or 'spring rate'. Dampers or shock absorbers are flexible, visco-elastic components, which means that their deformation is time-dependent. The force in a damper depends on the velocity of the applied deformation. Dampers also normally have some level of preload, which is a force and is not time dependent. Track bars and other links have very high stiffness and will be treated as rigid elements with very small, negligible, deformations under load.

The springs are always present in the suspension of a racing car. The other suspension components participate in transmitting loads, from the chassis to the wheels, at varying proportions depending on the material characteristics of the components (spring rate for the roll bar, shaft speed for the dampers) and the geometry of the suspension (location of Instant Centre, IC).

When two or more suspension components are present and active, *e.g.*

- spring + roll bar
- spring + damper
- spring + roll bar + damper

the total force, required for equilibrium, that is passed from the chassis to the wheels is split between the elements participating in the mechanism in proportion to the stiffness and geometry of the components.

If there are no other suspension components, the springs pass the total load from the chassis to the wheel. If there are active suspension components, they will take some of the total load and the spring will take the rest. This smaller spring load results in smaller spring deformations and lower values for chassis motion (squat, pitch and roll).

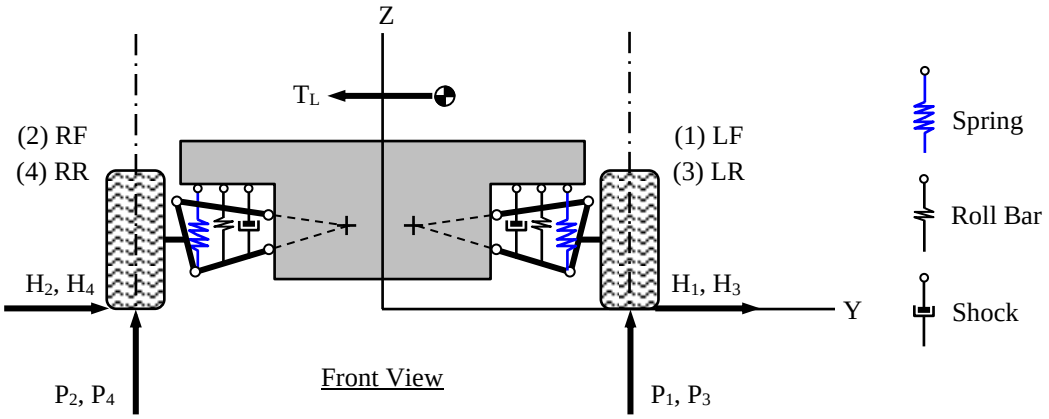
The amount of load the suspension elements are capable of developing depends on the geometry of the components, motion ratios, spring rates and location of the Instant Centre, IC.

Independent suspensions can be used at front or rear corners. For the purpose of this book, live axle track bar suspensions are used only at the rear. Roll bars are used at the front and/or rear. Dampers can also be used front and rear. Both independent and track bar suspensions are used in combination with roll bars and dampers. Springs are always present at all corners in all suspension mechanisms.

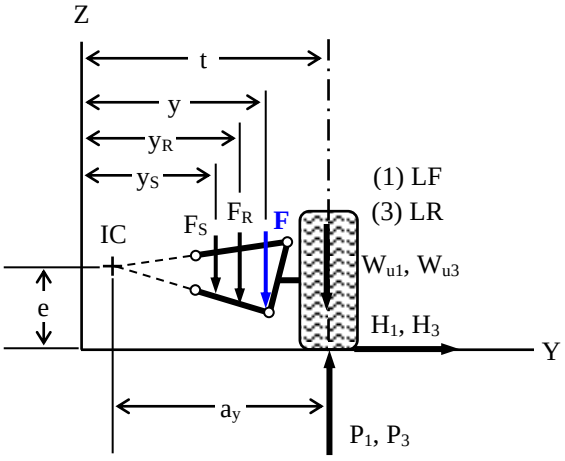
Independent Suspension

Under an independent suspension system, the horizontal lateral force induced by a turn produces a vertical reaction at the wheel contact patch, P, and a horizontal reaction H as shown. A portion of the total wheel load is passed to the chassis through the deformable components (spring, roll bar, shocks) and the rest is passed through the rigid links (upper and lower arms).

Left Turn



Corners (1) and (3). Left Turn



The horizontal reaction, H, to the turning force, T_L, depends on the portion contributed by each tire. The total $H_1 + H_2$ acting on the front tires is proportional to the amount of mass on the front tires. This portion can be divided between the front tires by assuming each contributes in proportion to the wheel load on the corner.

This can not be expressed in a linear equation so we assume the entire turning force is distributed in proportion to the total wheel load. This assumption is inaccurate when there is a longitudinal force or aerodynamic load.

$$H = \frac{P}{W_T} T_L \quad \rightarrow \quad H = \frac{T_L}{W_T} P$$

Equilibrium

$\boxed{\Sigma F_Z = 0}$ $\quad P - mF - m_R F_R - m_S F_S - w_u = 0$

$\Sigma M_{IC} = 0$

$$P a_y + H\, e - m F a_y - m_R F_R a_y - m_S F_S a_y - w_u\, a_y = 0$$

dividing by 'a_y'

$$P + H\,\frac{e}{a_y} - mF - m_R F_R - m_S F_S - w_u = 0$$

Spring, roll bar and damper behaviour

$$F = k\,(d + D + m_y R_C) \qquad F_R = k_R\,(d_R + D_R + m_R R_C) \qquad F_S = f_S + f_{SP}$$

Back substitution

$$P + \frac{T_L}{W_T} P\,\frac{e}{a_y} - m\,k\,(d + D + mR_C) - m_R\,k_R\left(\frac{m_R}{m} d + D_R\right) - m_S\,(f_S + f_{SP}) - w_u = 0$$

re-arranging and collecting terms

$$P\left[1 + \frac{T_L}{W_T}\,\frac{e}{a_y} - mkd - m_R k_R d_R\right] = mk(D + mR_C) + m_R k_R\,(D_R + m_R\,R_C) + m_S F_S + w_u$$

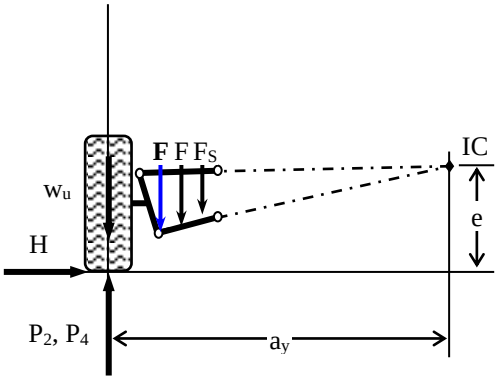
$$P\left[1 + \frac{T_L}{W_T}\,\frac{e}{a_y} - KV - m_R k_R d_R\right] = mk(D + mR_C) + m_R k_R\,(D_R + m_R\,R_C) + m_S F_S + w_u$$

Individual corners

$$P_1\left[1 + \frac{T_L}{W_T}\left(\frac{e}{a_Y}\right)_1\right] - K_1 V_1 - m_{R1} k_{RF} d_R = m_1 k_1\,(D_1 + m_1 R_{C1}) + m_{R1} k_{RF}\,(D_{R1} + m_{R1}\,R_{C1}) + m_{S1} F_{S1} + w_u$$

$$P_3\left[1 + \frac{T_L}{W_T}\left(\frac{e}{a_Y}\right)_3\right] - K_3 V_3 - m_{R3} k_{RR} d_R = m_3 k_3\,(D_3 + m_3 R_{C3}) + m_{R2} k_{RR}\,(D_{R3} + m_{R3}\,R_{C3}) + m_{S3} F_{S3} + w_u$$

Corners (2) and (4). Left Turn



Equilibrium

$\Sigma M_{IC} = 0$

$$-P a_y + H\, e + m F a_y + m_R F_R a_y + m_S F_S a_y + w_u\, a_y = 0$$

dividing by 'a_y'

$$-P + H\,\frac{e}{a_y} + mF + m_R F_R + m_S F_S + w_u = 0$$

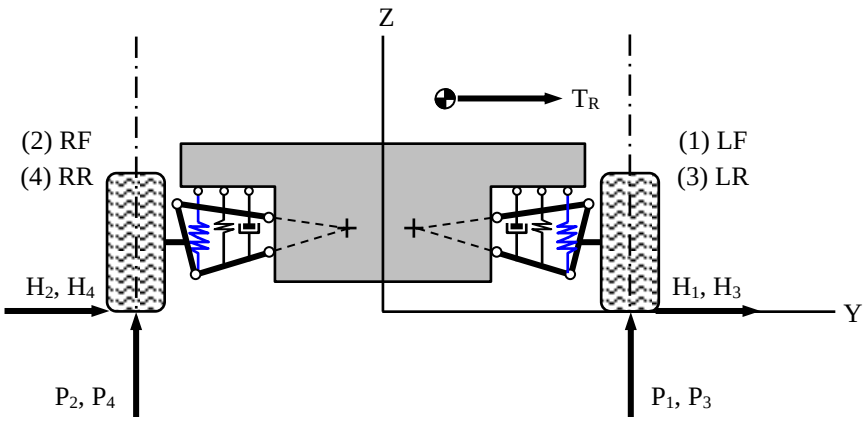
Using a similar algebraic procedure as for the left corner

Individual corners

$$P_2\left[1-\frac{T_L}{W_T}\left(\frac{e}{a_y}\right)_2\right]-K_2V_2-m_{R2}k_{RF}d_R=m_2k_2\,(D_2+m_2R_{C2})+m_{R2}k_{RF}\,(D_{R2}+m_{R2}\,R_{C2})+m_{S2}F_{S2}$$

$$P_4\left[1-\frac{T_L}{W_T}\left(\frac{e}{a_y}\right)_4\right]-K_4V_4-m_{R4}k_{RR}d_R=m_4k_4\,(D_4+m_4R_{C4})+m_{R4}k_{RR}\,(D_{R4}+m_{R4}\,R_{C4})+m_{S4}F_{S4}$$

Right Turn



Following the same steps as for the left turn case, with due to consideration to the change in sign of the horizontal forces, we obtain the following:

$$P_1\left[1+\frac{T_R}{W_T}\left(\frac{e}{a_Y}\right)_1\right]-K_1V_1-m_{R1}k_{RF}d_R=m_1k_1\,(D_1+m_1R_{C1})+m_{R1}k_{RF}\,(D_{R1}+m_{R1}\,R_{C1})+m_{S1}F_{S1}+w_{u1}$$

$$P_3\left[1+\frac{T_R}{W_T}\left(\frac{e}{a_y}\right)_3\right]-K_3V_3-m_{R3}k_{RR}d_R=m_3k_3\,(D_3+m_3R_{C3})+m_{R2}k_{RR}\,(D_{R3}+m_{R3}\,R_{C3})+m_{S3}F_{S3}+w_{u3}$$

$$P_2\left[1-\frac{T_R}{W_T}\left(\frac{e}{a_y}\right)_2\right]-K_2V_2-m_{R2}k_{RF}d_R=m_2k_2\,(D_2+m_2R_{C2})+m_{R2}k_{RF}\,(D_{R2}+m_{R2}\,R_{C2})+m_{S2}F_{S2}+w_{u2}$$

$$P_4\left[1-\frac{T_R}{W_T}\left(\frac{e}{a_y}\right)_4\right]-K_4V_4-m_{R4}k_{RR}d_R=m_4k_4\,(D_4+m_4R_{C4})+m_{R4}k_{RR}\,(D_{R4}+m_{R4}\,R_{C4})+m_{S4}F_{S4}+w_{u4}$$

Simplified Equations

These equations correspond to Equations [4] through [7] for the Base Case discussed before.

$$P_1\left[1+\frac{T_L+T_R}{W_T}\left(\frac{e}{a_Y}\right)_1\right]-K_1V_1-m_{R1}k_{RF}d_R=m_1k_1\,(D_1+m_1R_{C1})+m_{R1}k_{RF}\,(D_{R1}+m_{R1}\,R_{C1})+m_{S1}F_{S1}+w_{u1}$$

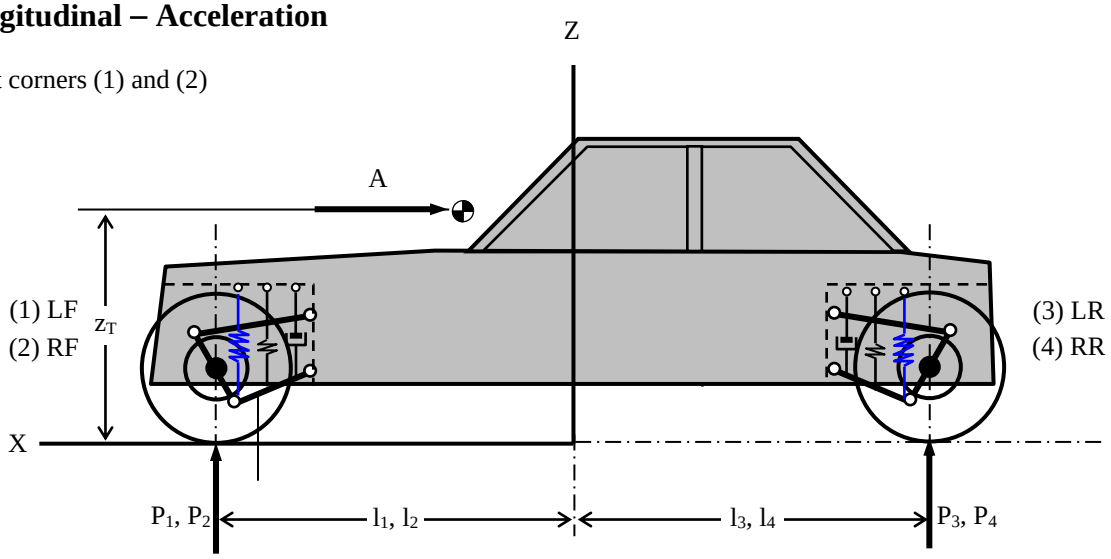
$$P_2\left[1-\frac{T_L+T_R}{W_T}\left(\frac{e}{a_y}\right)_2\right]-K_2V_2-m_{R2}k_{RF}d_R=m_2k_2\,(D_2+m_2R_{C2})+m_{R2}k_{RF}\,(D_{R2}+m_{R2}\,R_{C2})+m_{S2}F_{S2}+w_{u2}$$

$$P_3\left[1+\frac{T_L+T_R}{W_T}\left(\frac{e}{a_y}\right)_3\right]-K_3V_3-m_{R3}k_{RR}d_R=m_3k_3\,(D_3+m_3R_{C3})+m_{R2}k_{RR}\,(D_{R3}+m_{R3}\,R_{C3})+m_{S3}F_{S3}+w_{u3}$$

$$P_4\left[1-\frac{T_L+T_R}{W_T}\left(\frac{e}{a_y}\right)_4\right]-K_4V_4-m_{R4}k_{RR}d_R=m_4k_4\,(D_4+m_4R_{C4})+m_{R4}k_{RR}\,(D_{R4}+m_{R4}\,R_{C4})+m_{S4}F_{S4}+w_{u4}$$

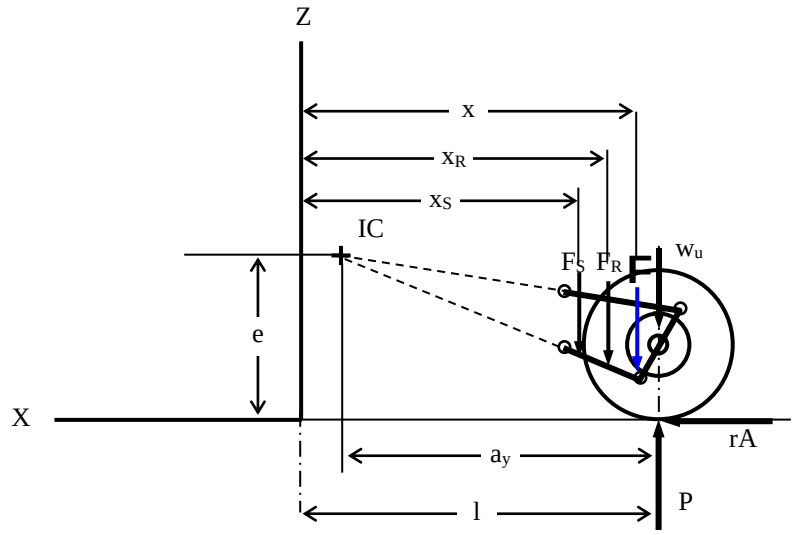
Longitudinal – Acceleration

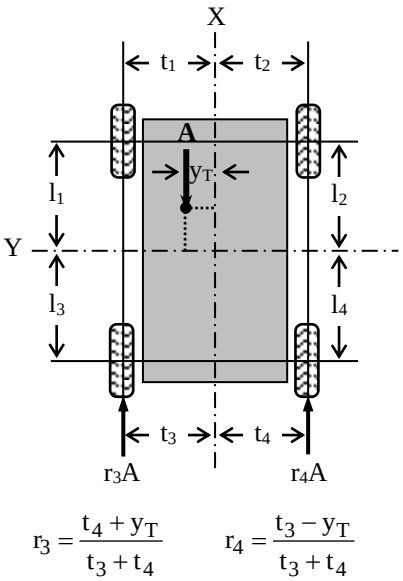
Front corners (1) and (2)



Rear corners (3) and (4)

The total acceleration force, **A**, is applied at the rear and split between corners (3) and (4) in proportion to the distance from the wheel patch to the CG





$$r_3 = \frac{t_4 + y_T}{t_3 + t_4} \qquad r_4 = \frac{t_3 - y_T}{t_3 + t_4}$$

$$P - rA\,\tfrac{e}{a_x} - mF - m_R F_R - m_S F_S - w_u = 0$$

$$P - rA\,\tfrac{e}{a_x} - mk\,(d + D + mR_C) - m_R k_R(d_R + D_R + m_R R_C) - m_S\,(f_S + f_{SP}) - w_u = 0$$

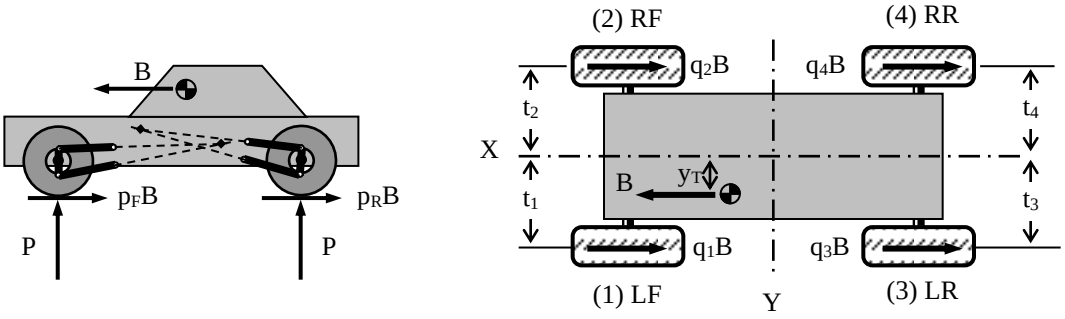
Individual corners and replacing d with V

$$P_3 - K_3V_3 - m_{R3}k_{RR}d_R = r_3\,A\left(\tfrac{e}{a_x}\right)_3 + m_3k_3\,(D_3 + m_3R_{C3}) + m_{R3}\,k_{RR}(D_{R3} + m_{R3}R_{C3}) + m_{S3}\,F_{S3} + w_{u3}$$

$$P_4 - K_4V_4 - m_{R4}k_{RR}d_R = r_4\,A\left(\tfrac{e}{a_x}\right)_4 + m_4k_4\,(D_4 + m_4R_{C4}) + m_{R4}\,k_{RR}(D_{R4} + m_{R4}R_{C4}) + m_{S4}\,F_{S4} + w_{u4}$$

Longitudinal – Braking

The total braking force, B, is split between the rear and the front wheel assemblies in the ratio $p_F : p_R$ The front and rear braking force components, $p_F B$ and $p_R B$, are in turn split between the left and right tires, in proportion to their distance to the CG.

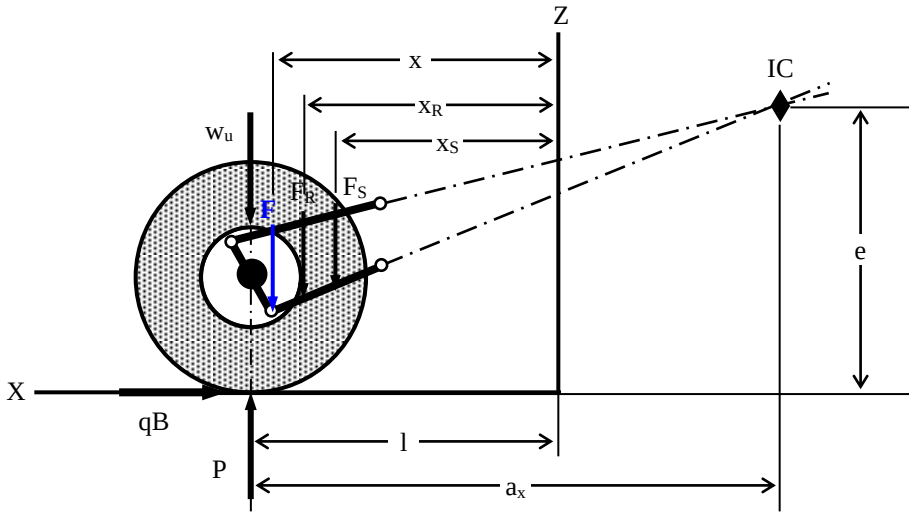


Percentage splits between front and rear, p_F and p_R, are assumed known. The left/right split is calculated as shown below.

$$q_1 = p_F \frac{t_2 + y_T}{t_1 + t_2} \qquad q_2 = p_F \frac{t_1 - y_T}{t_1 + t_2}$$

$$q_3 = p_R \frac{t_4 + y_T}{t_3 + t_4} \qquad q_4 = p_R \frac{t_3 - y_T}{t_3 + t_4}$$

Front corners (1) and (2)

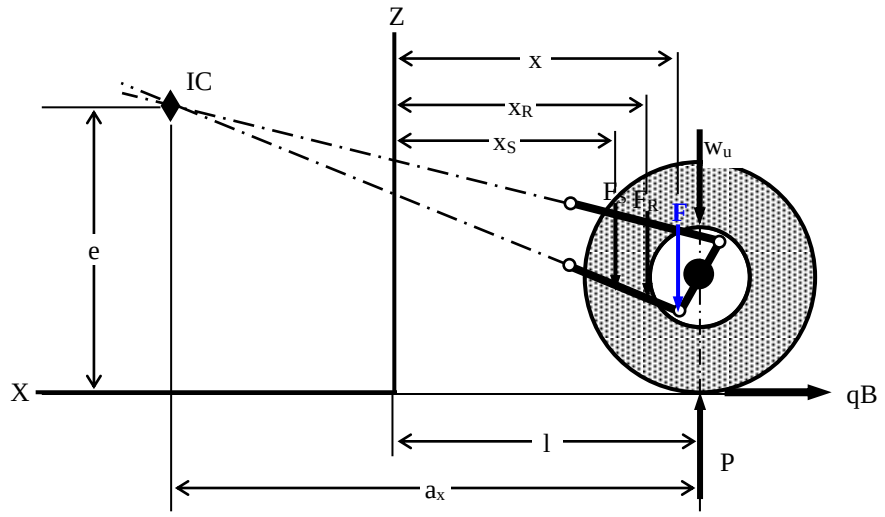


$$P - qB\frac{e}{a_x} - mF - m_R F_R - m_S F_S - w_u = 0$$

$$P_1 - K_1V_1 - m_{R1}k_{RF}d_R = q_1B\left(\frac{e}{a_x}\right)_1 + m_1k_1(D_1 + m_1R_{C1}) + m_{R1}k_{RF}(D_{R1} + m_{R1}R_{C1}) + m_{S1}F_{S1} + w_{u1}$$

$$P_2 - K_2V_2 - m_{R2}k_{RF}d_R = q_2B\left(\frac{e}{a_x}\right)_2 + m_2k_2(D_2 + m_2R_{C2}) + m_{R2}k_{RF}(D_{R2} + m_{R2}R_{C2}) + m_{S2}F_{S2} + w_{u2}$$

Rear Corners (3) and (4)



$$P + qB\frac{e}{a_x} - mF - m_R F_R - m_S F_S - w_u = 0$$

$$P_3 - K_3V_3 - m_{R3}k_{RR}d_R = -q_3B\left(\frac{e}{a_x}\right)_3 + m_3k_3(D_3 + m_3R_{C3}) + m_{R3}k_{RR}(D_{R3} + m_{R3}R_{C3}) + m_{S3}F_{S3} + w_{u3}$$

$$P_4 - K_4V_4 - m_{R4}k_{RR}d_R = -q_4B\left(\frac{e}{a_x}\right)_4 + m_4k_4(D_4 + m_4R_{C4}) + m_{R4}k_{R4}(D_{R4} + m_{R4}R_{C4}) + m_{S4}F_{S4} + w_{u4}$$

$$q_3 = p_R\frac{t_4 + y_T}{t_3 + t_4} \qquad q_4 = p_R\frac{t_3 - y_T}{t_3 + t_4}$$

Summary (Brake)

$$P_1 - K_1V_1 - m_{R1}k_{RF}d_R = q_1B\left(\frac{e}{a_x}\right)_1 + m_1k_1(D_1 + m_1R_{C1}) + m_{R1}k_{RF}(D_{R1} + m_{R1}R_{C1}) + m_{S1}F_{S1} + w_{u1}$$

$$P_2 - K_2V_2 - m_{R2}k_{RF}d_R = q_2B\left(\frac{e}{a_x}\right)_2 + m_2k_2(D_2 + m_2R_{C2}) + m_{R2}k_{RF}(D_{R2} + m_{R2}R_{C2}) + m_{S2}F_{S2} + w_{u2}$$

$$P_3 - K_3V_3 - m_{R3}k_{RR}d_R = -q_3B\left(\frac{e}{a_x}\right)_3 + m_3k_3(D_3 + m_3R_{C3}) + m_{R3}k_{RR}(D_{R3} + m_{R3}R_{C3}) + m_{S3}F_{S3} + w_{u3}$$

$$P_4 - K_4V_4 - m_{R4}k_{RR}d_R = -q_4B\left(\frac{e}{a_x}\right)_4 + m_4k_4(D_4 + m_4R_{C4}) + m_{R4}k_{R4}(D_{R4} + m_{R4}R_{C4}) + m_{S4}F_{S4} + w_{u4}$$

General Case. Independent Suspension at all corners. (All Loads, Banking, Antis)

Introduction

Previously, we have considered a Base Case with vertical force W_T, turning forces T_L and T_R, road banking through an angle α, acceleration force A, braking force B and other loads such as Ballast, Driver, Fuel and Aerodynamic. In this Base Case, we assumed that only the flexible suspension components (springs, roll bars and dampers) are active, ignoring the effect of the rigid components. The contribution of the rigid components to resist the loads passed from the contact patch to the chassis, and vice versa, depends on the position of the Instant Centre (IC), defined by the geometry of the suspension links. In the previous case, we assumed that these loads are resisted entirely by the springs. In this General Case, we incorporate the effect of the rigid suspension components. The magnitude of this contribution is dictated by the position of the IC, given by its horizontal distance to the centre of the contact patch, a, and its elevation from the ground, e.

Analysis

The overall analysis of the system follows the same concepts presented for the Base Case. First, we establish the conditions for equilibrium: Vertical forces (Z–Axis), Moments about the X–axis and Moments about the Y–axis, equations [1], [2] and [3]. Second, we write the equations describing the behaviour of the suspension components, equations [4], [5], [6] and [7]. Third, we write the equation of compatibility of deformations, equation [8]. Then we write the equation establishing the compliance of the roll bar to the deformations of the springs, equation [9]. Equations [1], [2], [3] and [8] involve the overall behaviour of the system and are the same as for the Base Case. Equations [4], [5], [6] and [7] involve the behaviour of individual suspension assemblies at each corner and will be different from those developed for the Base case. The difference is the addition of terms corresponding to 'jacking' forces induced by the rigid components.

Equations

[1] $$P_1 + P_2 + P_3 + P_4 = W_T \cos\alpha + (T_L + T_R) \sin\alpha + (Ba + Dr + Fu) \cos\alpha + Ae$$

[2] $$P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = [-R]\{ (W_T \sin\alpha - (T_L + T_R) \cos\alpha)\, z_T - (W_T \cos\alpha + (T_L + T_R) \sin\alpha)\, y_T + (Ba\, z_B + Dr\, z_D + Fu\, z_F) \sin\alpha - (Ba\, y_B + Dr\, y_D + Fu\, y_F) \cos\alpha + Ae\, y_A\}$$

[3] $$P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = (W_T \cos\alpha + (T_L + T_R) \sin\alpha)\, x_T + (B-A)\, z_T + (Ba\, x_B + Dr\, x_D + Fu\, x_F) \cos\alpha + Ae\, x_A$$

[4] $$P_1 \left[1 + \frac{T_L + T_R}{W_T} \left(\frac{e}{a_y}\right)_1\right] - K_1V_1 - m_{R1}k_{RF}d_R = q_1B \left(\frac{e}{a_x}\right)_1 + m_1k_1 (D_1 + m_1R_{C1}) + m_{R1}k_{RF} (D_{R1} + m_{R1} R_{C1}) + m_{S1}F_{S1} + w_{u1}$$

[5] $$P_2 \left[1 - \frac{T_L + T_R}{W_T} \left(\frac{e}{a_y}\right)_2\right] - K_2V_2 + m_{R2}k_{RF}d_R = q_2B \left(\frac{e}{a_x}\right)_2 + m_2k_2 (D_2 + m_2R_{C2}) + m_{R2}k_{RF} (D_{R2} + m_{R2} R_{C2}) + m_{S2}F_{S2} + w_{u2}$$

[6] $$P_3 \left[1 + \frac{T_L + T_R}{W_T} \left(\frac{e}{a_y}\right)_3\right] - K_3V_3 - m_{R3}k_{RR}d_R = r_3 A \left(\frac{e}{a_x}\right)_3 - q_3B \left(\frac{e}{a_x}\right)_3 + m_3k_3 (D_3 + m_3R_{C3}) + m_{R2}k_{RR} (D_{R3} + m_{R3} R_{C3}) + m_{S3}F_{S3} + w_{u3}$$

[7] $$P_4 \left[1 - \frac{T_L + T_R}{W_T} \left(\frac{e}{a_y}\right)_4\right] - K_4V_4 + m_{R4}k_{RR}d_R = r_4 A \left(\frac{e}{a_x}\right)_4 - q_4B \left(\frac{e}{a_x}\right)_4 + m_4k_4 (D_4 + m_4R_{C4}) + m_{R4}k_{RR} (D_{R4} + m_{R4} R_{C4}) + m_{S4}F_{S4} + w_{u4}$$

[8] $$C_1 V_1 + C_2 V_2 + C_3 V_3 + C_4 V_4 = 0$$

[9] $$V_1 - V_2 - G\, d_R = 0$$

Matrix Formulation

This system of 8 equations and 8 unknowns can be formulated using matrix notation.

$$\mathbf{A}\,X = \mathbf{B} \quad \rightarrow \quad X = \mathbf{A^{-1}}\,\mathbf{B}$$

Where

A = Matrix of stiffness and geometry coefficients
$\mathbf{A^{-1}}$ = Inverse of matrix A
X = Vector of unknown variables
B = Vector of constant coefficients

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ t_1 & -t_2 & t_3 & -t_4 & 0 & 0 & 0 & 0 & 0 \\ l_1 & l_2 & -l_3 & -l_4 & 0 & 0 & 0 & 0 & 0 \\ E_1 & 0 & 0 & 0 & -K_1 & 0 & 0 & 0 & K_{R1} \\ 0 & E_2 & 0 & 0 & 0 & -K_2 & 0 & 0 & K_{R2} \\ 0 & 0 & E_3 & 0 & 0 & 0 & -K_3 & 0 & K_{R3} \\ 0 & 0 & 0 & E_4 & 0 & 0 & 0 & -K_4 & K_{R4} \\ 0 & 0 & 0 & 0 & C_1 & C_2 & C_3 & C_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -G \end{bmatrix}$$

where

$$E_1 = 1 + \frac{T_L + T_R}{W_T}\left(\frac{e}{a_y}\right)_1 \qquad E_2 = 1 - \frac{T_L + T_R}{W_T}\left(\frac{e}{a_y}\right)_2$$

$$E_3 = 1 + \frac{T_L + T_R}{W_T}\left(\frac{e}{a_y}\right)_3 \qquad E_4 = 1 - \frac{T_L + T_R}{W_T}\left(\frac{e}{a_y}\right)_4$$

$K_i = m_i^2\, k_i$ for i = 1 … 4

$K_{R1} = -m_{R1}k_{RF}$ $K_{R2} = m_{R2}k_{RF}$ $K_{R3} = -m_{R3}k_{RR}$ $K_{R4} = m_{R4}k_{RR}$

C_i = Compatibility coefficients $G = \dfrac{y_1 + y_2}{y_{R1} + y_{R2}}$

$$X = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ d_R \end{Bmatrix} \quad \text{and} \quad B = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \\ B_9 \end{Bmatrix}$$

Constant coefficients, B, on the right side of the equations are:

$B_1 = W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha + (Ba + Dr + Fu) \text{ Cos } \alpha + Ae$

$B_2 = [-R]\{ (W_T \text{ Sin } \alpha - (T_L + T_R) \text{ Cos } \alpha) z_T - (W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha) y_T$

$(Ba\ z_B + Dr\ z_D + Fu\ z_F) \text{ Sin } \alpha - (Ba\ y_B + Dr\ y_D + Fu\ y_F) \text{ Cos } \alpha + Ae\ y_A \}$

$B_3 = (W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha)\ x_T + (B-A)\ z_T + (Ba\ x_B + Dr\ x_D + Fu\ x_F) \text{ Cos } \alpha + Ae\ x_A$

$B_4 = q_1 B \left(\frac{e}{a_x}\right)_1 + m_1 k_1 (D_1 + m_1 R_{C1}) + m_{R1} k_{RF} (D_{R1} + m_{R1} R_{C1}) + m_{S1} F_{S1} + w_{u1}$

$B_5 = q_2 B \left(\frac{e}{ax}\right)_2 + m_2 k_2 (D_2 + m_2 R_{C2}) + m_{R2} k_{RF} (D_{R2} + m_{R2} R_{C2}) + m_{S2} F_{S2} + w_{u2}$

$B_6 = r_3 A \left(\frac{e}{a_x}\right)_3 - q_3 B \left(\frac{e}{a_x}\right)_3 + m_3 k_3 (D_3 + m_3 R_{C3}) + m_{R3} k_{RR} (D_{R3} + m_{R3} R_{C3}) + m_{S3} F_{S3} + w_{u3}$

$B_7 = = r_4 A \left(\frac{e}{a_x}\right)_4 - q_4 B \left(\frac{e}{a_x}\right)_4 + m_4 k_4 (D_4 + m_4 R_{C4}) + m_{R4} k_{RR} (D_{R4} + m_{R4} R_{C4}) + m_{S4} F_{S4} + w_{u4}$

$B_8 = 0$

$B_9 = 0$

Where

$$r_3 = \frac{t_4 + y_T}{t_3 + t_4} \qquad r_4 = \frac{t_3 - y_T}{t_3 + t_4}$$

$$q_1 = p_F \frac{t_2 + y_T}{t_1 + t_2} \qquad q_2 = p_F \frac{t_1 - y_T}{t_1 + t_2}$$

$$q_3 = p_R \frac{t_4 + y_T}{t_3 + t_4} \qquad q_4 = p_R \frac{t_3 - y_T}{t_3 + t_4}$$

Solving this system of equations results in 4 wheel loads, 4 wheel motions and one roll bar distortion.

$P_1, P_2, P_3, P_4, V_1, V_2, V_3, V_4, d_R$

The wheel loads are used to calculate left, rear and wedge bias (B_L, B_R, B_W).

$$B_L = \frac{P_1 + P_3}{W_T} \qquad B_R = \frac{P_3 + P_4}{W_T} \qquad B_w = \frac{P_2 + P_3}{W_T}$$

With known geometry (spring co-ordinates x and y), the spring deformations are used to determine the co-ordinates of midpoints A, B, C and D.

Mid-Point Co-ordinates

	A	**B**	**C**	**D**
x	$\frac{y_1x_2 + y_2x_1}{y_1 + y_2}$	0	$-\frac{y_3x_4 + y_4x_3}{y_3 + y_4}$	0
y	0	$-\frac{x_2y_4 + x_4y_2}{x_2 + x_4}$	0	$\frac{x_1y_3 + x_3y_1}{x_1 + x_3}$
z	$\frac{y_1V_2 + y_2V_1}{y_1 + y_2}$	$\frac{x_2V_4 + x_4V_2}{x_2 + x_4}$	$\frac{y_3V_4 + y_4V_3}{y_3 + y_4}$	$\frac{x_1V_3 + x_3V_1}{x_1 + x_3}$

Note: x_i and y_i are the co-ordinates of tire contact points and V_i are the vertical wheel movements

These midpoint co-ordinates will be used to find the equation for the plane of the spring points.

$$AAx + BBy + CCz + DD = 0$$

Where

$$AA = y_1 V_2 + y_2 V_1 + y_C (V_1 - V_2) - z_C(y_1 + y_2)$$

$$BB = - x_1V_2 + x_2V_1 - x_C (V_1 - V_2) + z_C(x_1 - x_2)$$

$$CC = - x_1y_2 - x_2y_1 + x_C (y_1 + y_2) + y_C(x_2 - x_1)$$

$$DD = x_C (-y_1V_2 - y_2V_1) + y_C (x_1V_2 - x_2 V_1) + z_C (y_1 x_2 + y_2 x_1)$$

The equation of the plane is used to determine ride changes at the ride points. We substitute the co-ordinates x_r and y_r for each ride point into the equation of the plane to find its change in elevation, z_r.

$$z_r = \frac{-DD - AAx_r - BBy_r}{CC}$$

Wheel motions are also used to find vertical displacements at the midpoints, d_A, d_B, d_C, d_D, for a given geometry, and then to determine the chassis motion components: squat, pitch and roll. First we find the change in ride for the origin (x = 0 and y = 0)

$$V_0 = z_0 = \frac{-DD - AA(0) - BB(0)}{CC} = -\frac{DD}{CC}$$

then we find the change in elevation for midpoints A and B

$$d_A = z_A = \frac{-DD - AA(x_A) - BB(y_A)}{CC} \qquad \text{and} \qquad d_B = z_B = \frac{-DD - AA(x_B) - BB(y_B)}{CC}$$

We then find the motion components

$$\textit{Squat} \rightarrow V_0 \qquad \textit{Pitch} \rightarrow p = \frac{d_A - d_0}{x_A} \qquad \textit{Roll} \rightarrow r = \frac{d_B - d_0}{y_B}$$

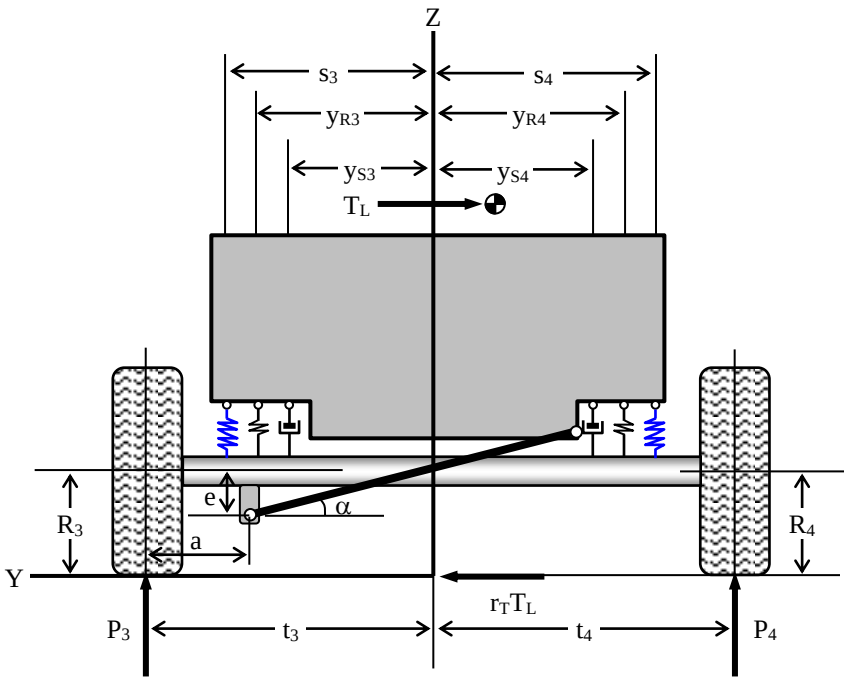
Live Axle Track Bar Suspension

In this case, the rear wheels are joined by a solid, rigid, axle and the front end has independent suspension at each corner. The rigid chassis is supported in the front by flexible components like springs, dampers and roll bars and a system of rigid links, and in the rear by flexible components connected to the live axle and by a single rigid track bar (panhard bar), as shown below.

The horizontal lateral force induced by a turn is transmitted from the chassis to the axle by the rigid track bar. The track bar is attached to the chassis at one end and to a point on the axle at the other. The force on the track bar induces 'anti' or jacking forces on the rear wheels with a magnitude that depends on the geometry of the suspension. Under these conditions, a portion of the total wheel loads is passed through the deformable components (spring, roll bar, shocks) and the rest is passed through the rigid track bar.

Lateral – Track bar (Panhard Bar)

This system consists of a solid rear axle with a bar attached to a point just below the axle at one end and to a point on the chassis at the opposite end. The analysis of the independent suspension system for the front wheels is explained in a previous section.



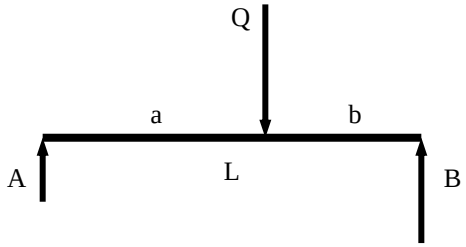
Rear View (YZ)

The turning force, T_L, is resisted by the front and rear wheels in proportion to the distance from the axle to the CG. Horizontal force $r_T T_L$ is the portion of the total turning force, T_L, resisted by the two rear tires at the contact patch. Both rear wheels are engaged in the mechanism, and, in this case, there is no need to separate this force into individual components under each wheel.

Analysis

The first problem is to determine the portion of the total lateral force acting on the rear tires. Since we are assuming the vehicle has no yaw moment, the lateral force acting on the rear tires is proportional to the mass on the rear tires.

First, some basic concepts: a rigid bar of length *L*, is acted upon by a single concentrated force, *Q*, at a distance *a* from the left support and *b* from the right support. The magnitude of the reaction at each support can be obtained by applying the moment equilibrium equation at each end, as shown below.



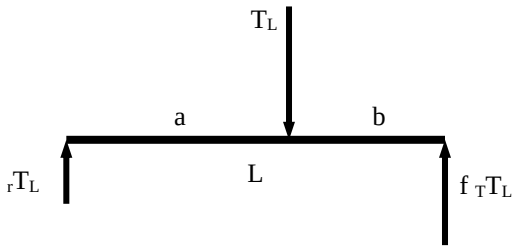
$\Sigma M_A = 0$

$$B\,L - Q\,a = 0 \rightarrow B = Q\frac{a}{L}$$

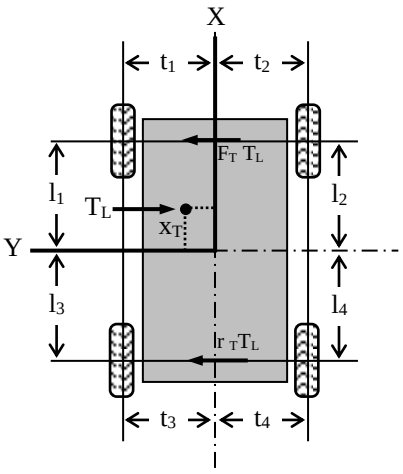
$\Sigma M_B = 0$

$$-A\,L + Q\,b = 0 \rightarrow A = Q\frac{b}{L}$$

In other words, the reaction at a given support is equal to the total load, divided by the total span and multiplied by the distance from the total load to the opposite support. We will use this concept to find the proportion of the turning force resisted by the rear tires. Consider a plan view of the chassis, under a turning load, applied at the CG, as shown below. Let $Q = T_L$, $A = r_T T_L$ and $B = f_T T_L$



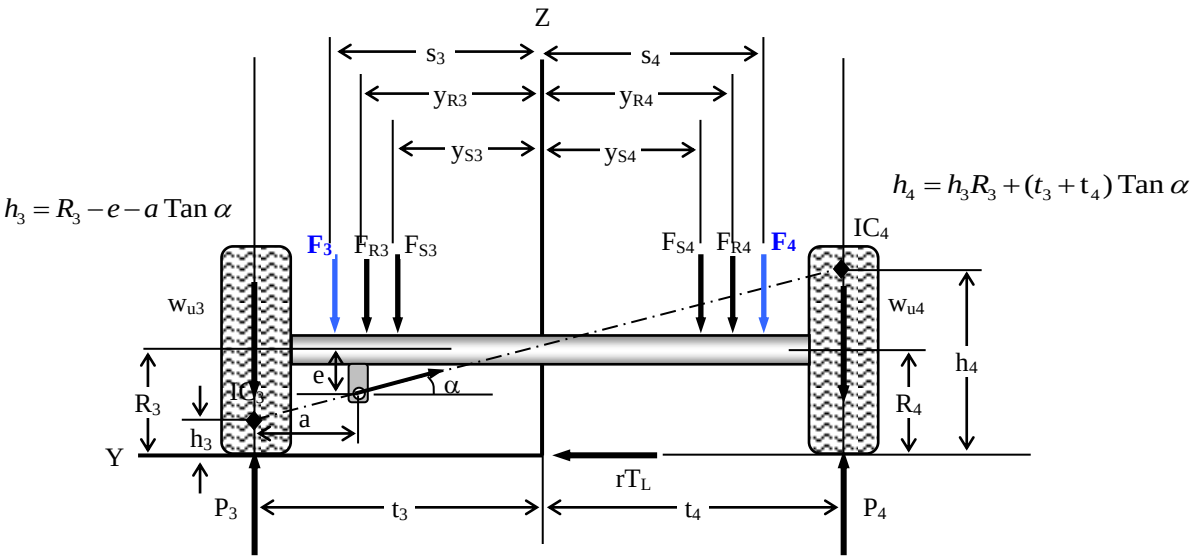
$$r_T T_L = T_L\frac{b}{L} \rightarrow r_T = \frac{b}{L}$$



Where

$$a = \frac{t_1 l_2 + t_2 l_1}{t_1 + t_2} \qquad b = \frac{t_3 l_4 + t_4 l_3}{t_3 + t_4} \qquad \rightarrow \qquad r_T = \frac{b}{a+b} =$$

Having determined the total lateral force acting on the rear axle, we can calculate the resulting forces.



Equilibrium

A double A-arm suspension is called “independent” because the movement and load on the right wheel is independent of the movement and load on the left wheel. The right wheel movement and load depends on the right spring deformation but not the left spring. The left wheel movement and load depends on the left spring deformation but not the right spring. A live axle is not independent because the movement and loads on either side depend on both springs. This makes the analysis more complicated.

To begin, we must determine the movement of the wheels and points s_3 and s_4 where the springs attach to the live axle. f_3 and f_4 are the vertical movements of s_3 and s_4. F_3 and F_4 are the forces applied at s_3 and s_4.

$$f_3 = V_3 \frac{s_3 + t_4}{t_3 + t_4} + V_4 \frac{t_3 - s_3}{t_3 + t_4}$$

$$f_4 = V_4 \frac{s_4 + t_3}{t_3 + t_4} + V_3 \frac{t_4 - s_4}{t_3 + t_4}$$

$d_3 = m_3 f_3$ because m_3 is the motion ratio between the spring and the attachment point on the rear axle. $m_3 = 1$ if the spring is attached directly to the rear axle.

Component behaviour

$F_3 = m_3 k_3 (d_3 + D_3) = m_3 k_3 (m_3 f_3 + D_3) = K_3 f_3 + m_3 k_3 D_3$ where $K_3 = m_3^2 k_3$
$F_4 = m_4 k_4 (d_4 + D_4) = m_4 k_4 (m_4 f_4 + D_4) = K_4 f_4 + m_4 k_4 D_4$ where $K_4 = m_4^2 k_4$

Here we consider the sum of moments about each of the two IC. We begin considering only the spring and spring preload. The other terms will be added later.

$\Sigma M_{IC4} = 0$

$$-P_3 (t_3 + t_4) - r_T T_L h_4 + F_3(s_3 + t_4) + F4(-s_4 + t_4) - w_{u3}(t_3 + t_4) = 0$$

dividing by $(t_3 + t_4)$ and re-arranging signs, we get:

$$P_3 + r_T T_L \frac{h_4}{t_3 + t_4} - F_3 \frac{s_3 + t_4}{t_3 + t_4} - F_4 \frac{-s_4 + t_4}{t_3 + t_4} - w_{u3} = 0$$

Moving some terms to the right

$$P_3 - F_3 \frac{s_3 + t_4}{t_3 + t_4} - F_4 \frac{-s_4 + t_4}{t_3 + t_4} = - r_T T_L \frac{h_4}{t_3 + t_4} + w_{u3}$$

Now replace F_3 with the equation using K_3 and f_3

$$P_3 - (K_3 f_3 + m_3 k_3 D_3) \frac{s_3 + t_4}{t_3 + t_4} - (K_4 f_4 + m_4 k_4 D_4) \frac{-s_4 + t_4}{t_3 + t_4} = - r_T T_L \frac{h_4}{t_3 + t_4} + w_{u3}$$

$$P_3 - K_3 f_3 \frac{s_3 + t_4}{t_3 + t_4} - K_4 f_4 \frac{-s_4 + t_4}{t_3 + t_4} = m_3 k_3 D_3 \frac{s_3 + t_4}{t_3 + t_4} + m_4 k_4 D_4 \frac{-s_4 + t_4}{t_3 + t_4} - r_T T_L \frac{h_4}{t_3 + t_4} + w_{u3}$$

Now replace f_3 and f_4 with equations involving V_3 and V_4

$$P_3 - K_3 \left(V_3 \frac{s_3 + t_4}{t_3 + t_4} + V_4 \frac{t_3 - s_3}{t_3 + t_4}\right) \frac{s_3 + t_4}{t_3 + t_4} - K_4 \left(V_4 \frac{s_4 + t_3}{t_3 + t_4} + V_3 \frac{t_4 - s_4}{t_3 + t_4}\right) \frac{-s_4 + t_4}{t_3 + t_4}$$
$$= m_3 k_3 D_3 \frac{s_3 + t_4}{t_3 + t_4} + m_4 k_4 D_4 \frac{-s_4 + t_4}{t_3 + t_4} - r_T T_L \frac{h_4}{t_3 + t_4} + w_{u3}$$

$$P_3 - L_3 V_3 - L_{3opp} V_4 = m_3 k_3 D_3 \frac{s_3 + t_4}{t_3 + t_4} + m_4 k_4 D_4 \frac{-s_4 + t_4}{t_3 + t_4} - r_T T_L \frac{h_4}{t_3 + t_4} + w_{u3}$$

Where $L_3 = K_3 \frac{s_3 + t_4}{t_3 + t_4} \frac{s_3 + t_4}{t_3 + t_4} + K_4 \frac{t_4 - s_4}{t_3 + t_4} \frac{-s_4 + t_4}{t_3 + t_4}$

And $L_{3opp} = K_3 \frac{t_3 - s_3}{t_3 + t_4} \frac{t_4 + s_3}{t_3 + t_4} + K_4 \frac{s_4 + t_3}{t_3 + t_4} \frac{t_4 - s_4}{t_3 + t_4}$

Now we take moments about IC_3

$\Sigma M_{IC3} = 0$

$$P_4 (t_3 + t_4) - r_T T_L h_3 - F_4(t_3 + s_4) - F_3(t_3 - s_3) - w_{u4}(t_3 + t_4) = 0$$

Dividing by ($t_3 + t_4$) and rearranging signs, we get:

$$P_4 - r_T T_L \frac{h_3}{t_3 + t_4} - F_4 \frac{t_3 + s_4}{t_3 + t_4} - - F_3 \frac{t_3 - s_3}{t_3 + t_4} - w_{u4} = 0$$

Moving some terms to the right

$$P_4 - F_4 \frac{t_3 + s_4}{t_3 + t_4} - F_3 \frac{t_3 - s_3}{t_3 + t_4} = r_T T_L \frac{h_4}{t_3 + t_4} + w_{u4}$$

Now replace F_4 with the equations using K_4 and f_4

$$P_4 - (K_4 f_4 + m_4 k_4 D_4) \frac{t_3 + s_4}{t_3 + t_4} - (K_3 f_3 + m_3 k_3 D_3) \frac{t_3 - s_3}{t_3 + t_4} = r_T T_L \frac{h3}{t_3 + t_4} + w_{u4}$$

$$P_4 - K_4 f_4 \frac{t_3 + s_4}{t_3 + t_4} - K_3 f_3 \frac{t_3 - s_3}{t_3 + t_4} = m_4 k_4 D_4 \frac{t_3 + s_4}{t_3 + t_4} + m_3 k_3 D_3 \frac{t_3 - s_3}{t_3 + t_4} + r_T T_L \frac{h_3}{t_3 + t_4} + w_{u4}$$

Now replace f_3 and f_4 with equations involving V_3 and V_4

$$P_4 - K_4 (V_4 \frac{s_4 + t_3}{t_3 + t_4} + V_3 \frac{t_4 - s_4}{t_3 + t_4}) \frac{t_3 + s_4}{t_3 + t_4} - K_3 (V_3 \frac{s_3 + t_4}{t_3 + t_4} + V_4 \frac{t_3 - s_3}{t_3 + t_4}) \frac{t_3 - s_3}{t_3 + t_4}$$
$$= m_4 k_4 D_4 \frac{t_3 + s_4}{t_3 + t_4} + m_3 k_3 D_3 \frac{t_3 - s_3}{t_3 + t_4} + r_T T_L \frac{h_3}{t_3 + t_4} + w_{u4}$$

$$P_4 - L_4 V_4 - L_{4opp} V_3 = m_4 k_4 D_4 \frac{t_3 + s_4}{t_3 + t_4} + m_3 k_3 d_3 \frac{t_3 - s_3}{t_3 + t_4} + r_T T_L \frac{h3}{t_3 + t_4} + w_{u4}$$

Where $L_4 = K_4 \frac{s_4 + t_3}{t_3 + t_4} \frac{t_3 + s_4}{t_3 + t_4} + K_3 \frac{t_3 - s_3}{t_3 + t_4} \frac{t_3 - s_3}{t_3 + t_4}$

And $L_{4opp} = K_4 \frac{t_4 - s_4}{t_3 + t_4} \frac{t_3 + s_4}{t_3 + t_4} + K_3 \frac{s_3 + t_4}{t_3 + t_4} \frac{t_3 - s_3}{t_3 + t_4}$

Now we add additional terms. First consider the shock force.
This operates just like spring preload except the motion ratio may be different and we use ys_3 and ys_4 in place of s_3 and s_4.

$$P_3 - L_3 V_3 - L_{3opp} V_4 = m_3 k_3 D_3 \frac{t_4 + s_3}{t_3 + t_4} + m_{s3} F_{s3} \frac{t_4 + y_{S3}}{t_3 + t_4} + m_4 k_4 D_4 \frac{-s_4 + t_4}{t_3 + t_4} + m_{s4} F_{s4} \frac{-y_{S4} + t_4}{t_3 + t_4} -$$
$$r_T T_L \frac{h_4}{t_3 + t_4} + w_{u3}$$

$$P_4 - L_4\,V_4 - L_{4opp}\,V_3 = m_4\,k_4\,D_4\,\frac{t_3 + s_4}{t_3 + t_4} + m_{s4}\,F_{s4}\,\frac{t_3 + y_{S4}}{t_3 + t_4} + m_3\,k_3\,d_3\,\frac{t_3 - s_3}{t_3 + t_4} + m_{s3}\,F_{s3}\,\frac{t_3 - y_{S3}}{t_3 + t_4} + r_T T_L$$

$$\frac{h3}{t_3 + t_4} + w_{u4}$$

Now consider radius change. This operates just like V3 and V4

$$P_3 - L_3\,(V_3 + R_{C3}) - L_{3opp}\,(V_4 + R_{C4}) = m_3\,k_3\,D_3\,\frac{t_4 + s_3}{t_3 + t_4} + m_{s3}\,F_{s3}\,\frac{t_4 + y_{S3}}{t_3 + t_4} + m_4\,k_4\,D_4\,\frac{-s_4 + t_4}{t_3 + t_4} + m_{s4}\,F_{s4}$$

$$\frac{-y_{S4} + t_4}{t_3 + t_4} - r_T T_L\,\frac{h_4}{t_3 + t_4} + w_{u3}$$

$$P_3 - L_3\,V_3 - L_{3opp}\,V_4 = L_3\,R_{C3} + L_{3opp}\,R_{C4} + m_3\,k_3\,D_3\,\frac{t_4 + s_3}{t_3 + t_4} + m_{s3}\,F_{s3}\,\frac{t_4 + y_{S3}}{t_3 + t_4} + m_4\,k_4\,D_4\,\frac{-s_4 + t_4}{t_3 + t_4} +$$

$$m_{s4}\,F_{s4}\,\frac{-y_{S4} + t_4}{t_3 + t_4} - r_T T_L\,\frac{h_4}{t_3 + t_4} + w_{u3}$$

$$P_4 - L_4\,(V_4 + Rc_4) - L_{4opp}\,(V_3 + Rc_3) = m_4\,k_4\,D_4\,\frac{t_3 + s_4}{t_3 + t_4} + m_{s4}\,F_{s4}\,\frac{t_3 + y_{S4}}{t_3 + t_4} + m_3\,k_3\,d_3\,\frac{t_3 - s_3}{t_3 + t_4} + m_{s3}\,F_{s3}$$

$$\frac{t_3 - y_{S3}}{t_3 + t_4} + r_T T_L\,\frac{h3}{t_3 + t_4} + w_{u4}$$

$$P_4 - L_4\,V_4 - L_{4opp}\,V_3 = L_4\,Rc_4 + L_{4opp}\,Rc_3 + m_4\,k_4\,D_4\,\frac{t_3 + s_4}{t_3 + t_4} + m_{s4}\,F_{s4}\,\frac{t_3 + y_{S4}}{t_3 + t_4} + m_3\,k_3\,d_3\,\frac{t_3 - s_3}{t_3 + t_4} + m_{s3}$$

$$F_{s3}\,\frac{t_3 - y_{S3}}{t_3 + t_4} + r_T T_L\,\frac{h3}{t_3 + t_4} + w_{u4}$$

The Anti Roll Bar (Swaybar) on a live axle works the same as with an independent suspension system except the motion ratio m_r is the ratio of the distance between the blades $y_{r3} + y_{r4}$ divided by the track or $t_3 + t_4$.

General Case
Front independent suspension. Solid Axle rear suspension.
(All Loads, Banking, antis)

Introduction

Previously, we considered the case of a car with independent suspension front and rear. In this case, we replace the independent suspension at the rear with a solid axle suspension with a track bar. As before, we incorporate the effect of the rigid suspension components.

Analysis

The overall analysis of the system follows the same concepts presented before. First, we establish the conditions for equilibrium: Vertical forces (Z), Moments about the X−axis and Moments about the Y−axis, equations [1], [2] and [3]. Second, we write the equations describing the behaviour of the suspension components, equations [4], [5], [6] and [7]. For this case equations [4] and [5] are the same as for the previous one because the front remains as an independent suspension system. Equations [6] and [7] for the rear axle are the ones developed for a track bar suspension. Third, we write the equation of compatibility of deformations, equation [8]. Then we write the equation establishing the compliance of the roll bar to the deformations of the springs, Equation [9]. Equations [1], [2], [3] and [8] involve the overall behaviour of the system and are the same as for the General Case with independent suspension in all corners. Equations [4] and [5], as indicated above, are also the same. Equations [6] and [7] involve the behaviour of the solid axle suspension in the rear and will be different from those developed for the previous case.

Equations

[1] $P_1 + P_2 + P_3 + P_4 = W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha + (Ba + Dr + Fu) \text{ Cos } \alpha + Ae$

[2] $P_1t_1 - P_2t_2 + P_3t_3 - P_4t_4 = [-R]\{ (W_T \text{ Sin } \alpha - (T_L + T_R) \text{ Cos } \alpha)\, z_T - (W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha)\, y_T$
$(Ba\, z_B + Dr\, z_D + Fu\, z_F) \text{ Sin } \alpha - (Ba\, y_B + Dr\, y_D + Fu\, y_F) \text{ Cos } \alpha + Ae\, y_A \}$

[3] $P_1l_1 + P_2l_2 - P_3l_3 - P_4l_4 = (W_T \text{ Cos } \alpha + (T_L + T_R) \text{ Sin } \alpha)\, x_T$
$+ (B-A)\, z_T + (Ba\, x_B + Dr\, x_D + Fu\, x_F) \text{ Cos } \alpha + Ae\, x_A$

[4] $P_1 \left[1 + \frac{T_L + T_R}{W_T} \left(\frac{e}{a_y}\right)_1\right] - K_1V_1 - m_{R1}k_{RF}d_R =$
$q_1B \left(\frac{e}{a_x}\right)_1 + m_1k_1 (D_1 + m_1R_{C1}) + m_{R1}k_{RF} (D_{R1} + m_{R1} R_{C1}) + m_{S1}F_{S1} + w_{u1}$

[5] $P_2 \left[1 - \frac{T_L + T_R}{W_T} \left(\frac{e}{a_y}\right)_2\right] - K_2V_2 + m_{R2}k_{RF}d_R =$
$q_2B \left(\frac{e}{a_x}\right)_2 + m_2k_2 (D_2 + m_2R_{C2}) + m_{R2}k_{RF} (D_{R2} + m_{R2} R_{C2}) + m_{S2}F_{S2} + w_{u2}$

[6] $P_3 - L_3 V_3 - L_{3opp} V_4 = m_3 k_3 D_3 \frac{t_4 + s_3}{t_3 + t_4} + m_{S3} F_{S3} \frac{t_4 + y_{S3}}{t_3 + t_4} + m_4 k_4 D_4 \frac{-s_4 + t_4}{t_3 + t_4} + m_{S4} F_{S4} \frac{-y_{S4} + t_4}{t_3 + t_4} -$
$r_T T_L \frac{h_4}{t_3 + t_4} + w_{u3}$

[7] $P_4 - L_4 V_4 - L_{4opp} V_3 = m_4 k_4 D_4 \frac{t_3 + s_4}{t_3 + t_4} + m_{S4} F_{S4} \frac{t_3 + y_{S4}}{t_3 + t_4} + m_3 k_3 d_3 \frac{t_3 - s_3}{t_3 + t_4} + m_{S3} F_{S3} \frac{t_3 - y_{S3}}{t_3 + t_4} + r_T T_L$
$\frac{h3}{t_3 + t_4} + w_{u4}$

[8] $C_1 V_1 + C_2 V_2 + C_3 V_3 + C_4 V_4 = 0$

[9] $V_1 - V_2 - G\, d_R = 0$

Matrix Formulation

This system of 9 equations and 9 unknowns can be formulated using matrix notation.

$$\mathbf{A}\,X = \mathbf{B} \quad \rightarrow \quad X = \mathbf{A^{-1}}\,\mathbf{B}$$

Where

A = Matrix of stiffness and geometry coefficients
$\mathbf{A^{-1}}$ = Inverse of matrix A
X = Vector of unknown variables (Four wheel loads, P, and four spring deformations, d)
B = Vector of constant coefficients

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ t_1 & -t_2 & t_3 & -t_4 & 0 & 0 & 0 & 0 & 0 \\ l_1 & l_2 & -l_3 & -l_4 & 0 & 0 & 0 & 0 & 0 \\ E_1 & 0 & 0 & 0 & -K_1 & 0 & 0 & 0 & K_{R1} \\ 0 & E_2 & 0 & 0 & 0 & -K_2 & 0 & 0 & K_{R2} \\ 0 & 0 & E_3 & 0 & 0 & 0 & -L_3 & -L_{3opp} & K_{R3} \\ 0 & 0 & 0 & E_4 & 0 & 0 & -L_{4opp} & -L_4 & K_{R4} \\ 0 & 0 & 0 & 0 & C_1 & C_2 & C_3 & C_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -G \end{bmatrix}$$

where

$E_1 = 1 + \frac{T_L + T_R}{W_T}\left(\frac{e}{a_y}\right)_1$ $\qquad$ $E_2 = 1 - \frac{T_L + T_R}{W_T}\left(\frac{e}{a_y}\right)_2$

$K_i = m_i^2\, k_I$ $\qquad$ for i = 1 … 4

$K_{R1} = -m_{R1}k_{RF}$ $\quad$ $K_{R2} = m_{R2}k_{RF}$ $\quad$ $K_{R3} = -m_{R3}k_{RR}$ $\quad$ $K_{R4} = m_{R4}k_{RR}$

C_i = Compatibility coefficients $\qquad$ $G = \dfrac{y_1 + y_2}{y_{R1} + y_{R2}}$

$$X = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ d_R \end{Bmatrix} \quad \text{and} \quad B = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \\ B_9 \end{Bmatrix}$$

Constant coefficients, B, on the right side of the equations are:

$$B_1 = W_T \cos\alpha + (T_L + T_R) \sin\alpha + (Ba + Dr + Fu) \cos\alpha + Ae$$

$$B_2 = [-R]\{ (W_T \sin\alpha - (T_L + T_R) \cos\alpha) z_T - (W_T \cos\alpha + (T_L + T_R) \sin\alpha) y_T$$
$$(Ba\, z_B + Dr\, z_D + Fu\, z_F + Ae\, z_A) \sin\alpha - (Ba\, y_B + Dr\, y_D + Fu\, y_F) \cos\alpha + Ae\, y_A \}$$

$$B_3 = (W_T \cos\alpha + (T_L + T_R) \sin\alpha)\, x_T + (B-A)\, z_T + (Ba\, x_B + Dr\, x_D + Fu\, x_F) \cos\alpha + Ae\, x_A$$

$$B_4 = q_1 B \left(\frac{e}{a_x}\right)_1 + m_1 k_1 (D_1 + m_1 R_{C1}) + m_{R1} k_{RF} (D_{R1} + m_{R1} R_{C1}) + m_{S1} F_{S1} + w_{u1}$$

$$B_5 = q_2 B \left(\frac{e}{a_x}\right)_2 + m_2 k_2 (D_2 + m_2 R_{C2}) + m_{R2} k_{RF} (D_{R2} + m_{R2} R_{C2}) + m_{S2} F_{S2} + w_{u2}$$

$$B_6 = m_3 k_3 D_3 \frac{t_4 + s_3}{t_3 + t_4} + m_{S3} F_{S3} \frac{t_4 + y_{S3}}{t_3 + t_4} + m_4 k_4 D_4 \frac{-s_4 + t_4}{t_3 + t_4} + m_{S4} F_{S4} \frac{-y_{S4} + t_4}{t_3 + t_4} - r_T T_L \frac{h_4}{t_3 + t_4} + w_{u3}$$

$$B_7 = m_4 k_4 D_4 \frac{t_3 + s_4}{t_3 + t_4} + m_{S4} F_{S4} \frac{t_3 + y_{S4}}{t_3 + t_4} + m_3 k_3 d_3 \frac{t_3 - s_3}{t_3 + t_4} + m_{S3} F_{S3} \frac{t_3 - y_{S3}}{t_3 + t_4} + r_T T_L \frac{h3}{t_3 + t_4} + w_{u4}$$

$$B_8 = 0$$

$$B_9 = 0$$

Where

$$r_3 = \frac{t_4 + y_T}{t_3 + t_4} \qquad r_4 = \frac{t_3 - y_T}{t_3 + t_4}$$

$$q_1 = p_F \frac{t_2 + y_T}{t_1 + t_2} \qquad q_2 = p_F \frac{t_1 - y_T}{t_1 + t_2}$$

$$q_3 = p_R \frac{t_4 + y_T}{t_3 + t_4} \qquad q_4 = p_R \frac{t_3 - y_T}{t_3 + t_4}$$

$$r_T = \frac{b}{a + b} \quad \rightarrow \quad a = \frac{t_1 l_2 + t_2 l_1}{t_1 + t_2} \text{ and } b = \frac{t_3 l_4 + t_4 l_3}{t_3 + t_4}$$

$$h_3 = R_3 - e - a \tan\alpha \qquad h_4 = h_3 R_3 + (t_3 + t_4) \tan\alpha$$

Solving this system of equations results in 4 wheel loads, 4 WHEEL MOTIONS and one roll bar distortion.

$$P_1, P_2, P_3, P_4, V_1, V_2, V_3, V_4, d_R$$

The wheel loads are used to calculate left, rear and wedge bias (B_L, B_R, B_W).

$$B_L = \frac{P_1 + P_3}{W_T} \qquad B_R = \frac{P_3 + P_4}{W_T} \qquad B_w = \frac{P_2 + P_3}{W_T}$$

With known geometry (spring co-ordinates x and y), the spring deformations are used to determine the co-ordinates of midpoints A, B, C and D.

Mid-Point Co-ordinates

	A	**B**	**C**	**D**
x	$\frac{y_1x_2 + y_2x_1}{y_1 + y_2}$	0	$-\frac{y_3x_4 + y_4x_3}{y_3 + y_4}$	0
y	0	$-\frac{x_2y_4 + x_4y_2}{x_2 + x_4}$	0	$\frac{x_1y_3 + x_3y_1}{x_1 + x_3}$
z	$\frac{y_1V_2 + y_2V_1}{y_1 + y_2}$	$\frac{x_2V_4 + x_4V_2}{x_2 + x_4}$	$\frac{y_3V_4 + y_4V_3}{y_3 + y_4}$	$\frac{x_1V_3 + x_3V_1}{x_1 + x_3}$

Note: x_i and y_i are the co-ordinates of tire contact points and V_i are the vertical wheel movements

These midpoint co-ordinates will be used to find the equation for the plane of the spring points.

$$AAx + BBy + CCz + DD = 0$$

Where

$$AA = y_1 V_2 + y_2 V_1 + y_C (V_1 - V_2) - z_C(y_1 + y_2)$$

$$BB = - x_1V_2 + x_2V_1 - x_C (V_1 - V_2) + z_C(x_1 - x_2)$$

$$CC = - x_1y_2 - x_2y_1 + x_C (y_1 + y_2) + y_C(x_2 - x_1)$$

$$DD = x_C (-y_1V_2 - y_2V_1) + y_C (x_1V_2 - x_2 V_1) + z_C (y_1 x_2 + y_2 x_1)$$

The equation of the plane is used to determine ride changes at the ride points. We substitute the co-ordinates x_r and y_r for each ride point into the equation of the plane to find its change in elevation, z_r.

$$z_r = \frac{-DD - AAx_r - BBy_r}{CC}$$

Wheel motions are also used to find vertical displacements at the midpoints, d_A, d_B, d_C, d_D, for a given geometry, and then to determine the chassis motion components: squat, pitch and roll. First we find the change in ride for the origin (x = 0 and y = 0)

$$V_0 = z_0 = \frac{-DD - AA(0) - BB(0)}{CC} = -\frac{DD}{CC}$$

then we find the change in elevation for midpoints A and B

$$d_A = z_A = \frac{-DD - AA(x_A) - BB(y_A)}{CC} \qquad \text{and} \qquad d_B = z_B = \frac{-DD - AA(x_B) - BB(y_B)}{CC}$$

We then find the motion components

$$\textit{Squat} \rightarrow V_0 \qquad \textit{Pitch} \rightarrow p = \frac{d_A - V_0}{x_A} \qquad \textit{Roll} \rightarrow r = \frac{d_B - V_0}{y_B}$$

Notation and Symbols

a	Factor used to calculate the distribution of turning force T_L to the rear axle
a_x	Horizontal distance from the wheel load to the Instant Centre (IC) in the longitudinal (X) direction
a_y	Horizontal distance from the wheel load to the Instant Centre (IC) in the lateral (Y) direction
A	Midpoint between LF and RF springs
A	Matrix of coefficients for model equations
AA	Coefficient of the x-co-ordinate in the spring plane equation
Ae	Aerodynamic load
α	Angle of bank of the road
α	Angle of the track bar in solid axle suspension
b	Factor used to calculate the distribution of turning force T_L to the rear axle
B	Coefficient used to calculate the distribution of turning force T_L to the rear axle
B	Midpoint between RF and RR springs
B	Braking force
B	Vector of constants for the model equations
Ba	Ballast load
BB	Coefficient of the y-co-ordinate in the spring plane equation
B_L	Left bias. Left wheel loads divided by total weight.
B_R	Rear Bias. Rear wheel loads divided by total weight
B_W	Wedge. RF plus LR wheel loads divided by total weight
C	Midpoint between LR and RR springs
C_i	Compatibility coefficients applied to spring deformations
CC	Coefficient of the z-co-ordinate in the spring plane equation
CG	Centre of gravity
d	Spring deformation
d_i	Spring deformation for i = 1 … 4
d_R	Roll bar distortion
d_{Ri}	Roll bar blade tip displacement for i = 1 … 4
D	Midpoint between LF and LR springs
D_i	Spring preload for i = 1 … 4
Dr	Driver load
D_{Ri}	Roll bar preload for i = 1 … 4
DD	Independent constant coefficient in the spring plane equation
e	Elevation of the Instant Centre (IC)
E_i	Wheel load coefficient to account for jacking forces in independent suspension for i = 1 … 4
f_{Si}	Force on the shock for i = 1 … 4
f_{SPi}	Shock pre load for i = 1 … 4
F_i	Spring force for i = 1 … 4
F_R	Force at the tip of the blade of a roll bar
F_{Si}	Total force on the shock for i = 1 … 4
Fu	Fuel load
G	Ratio of spring track to roll bar track. Geometric coefficient relating spring and roll bar deformations
h_3	Elevation of the IC at the LR wheel in a track bar suspension
h_4	Elevation of the IC at the RR wheel in a track bar suspension
H	Horizontal wheel force induced by a turn
IC	Instant Centre. Located at the intersection of the upper and lower control arm lines of action.
k	Spring rate or stiffness
k_I	Spring rate or stiffness for i = 1 … 4
k_R	Roll bar rate or stiffness

k_{RF} Front end roll bar rate or stiffness
k_{RR} Rear end roll bar rate or stiffness
K_i Spring stiffness coefficient for i = 1 … 4. [$K_i = m_i^2 k_i$]. Used in matrix **A**
K_R Roll bar stiffness coefficient $K_L mi^2 K^2$ for i = 1 … 4. [$K_{Ri} = m_{Ri} k_{Ri}$]. Used in matrix **A**

l Wheel base
l_i Corner wheel base for i = 1 … 4
L Force on Lower control arm
LF Left Front corner, also designated (1)
LR Left Rear corner, also designated (3)
L_3 Coefficient for solid axle suspension used in Matrix A
L_{3opp} Coefficient for solid axle suspension used in Matrix A
L_4 Coefficient for solid axle suspension used in Matrix A
L_{4opp} Coefficient for solid axle suspension used in Matrix A

m Spring motion ratio. Ratio between spring force and wheel load. P = mF
m_i Spring motion ratio for i = 1 … 4
m_{Ri} Roll bar motion ratio for i = 1 … 4. Ratio between RB force and wheel load. $P = m_R F_R$
m_{Si} Shock motion ratio for i = 1 … 4. Ratio between damper force and wheel load. $P = m_S F_S$

p Pitch
p_F Percent front brakes
p_R Percent rear brakes
P_i Wheel load for i = 1 … 4

q_1 Percent brake force to LF wheel
q_2 Percent brake force to RF wheel
q_3 Percent brake force to LR wheel
q_4 Percent brake force to RR wheel

r Roll
r_3 Percent acceleration force to LR wheel
r_4 Percent acceleration force to RR wheel
r_T Percent turn force ($T_L + T_R$) to rear end
[–R] Use negative sign for Right turn
R_i Wheel radius i = 1 … 4
R_{Ci} Change in wheel radius i = 1 … 4
RF Right Front corner, also designated (2)
RR Right Rear corner, also designated (3)

t Track
t_i Corner track i = 1 … 4
T_L Force induced by a left turn
T_R Force induced by a right turn

U Force on upper control arm
Vo Squat. Vertical displacement of the origin of co-ordinates (car centre).
V_1 Wheel motion LF
V_2 Wheel motion RF
V_3 Wheel motion LR
V_4 Wheel motion RR

w_u Unsprung weight for an individual wheel assembly for i = 1 … 4
W_S Sprung weight.
W_T Total weight of the car
W_U Total unsprung weight

x_{ri} X co-ordinate of ride point for i = 1 … 4

x_A X co-ordinate for midpoint A between LF and RF springs
x_{Ae} X co-ordinate for Aerodynamic load
x_B X co-ordinate for midpoint B between RF and RR springs
x_{Ba} X co-ordinate for Ballast load
x_C X co-ordinate for midpoint C between LR and RR springs
x_D X co-ordinate for midpoint D between LF and LR springs
x_{Dr} X co-ordinate for Driver load
x_{Fu} X co-ordinate for Fuel load
x_{Ri} X co-ordinate for roll bar bottom point for i = 1 … 4
x_{Si} X co-ordinate for shock bottom point for i = 1 … 4
x_S X co-ordinate of CG for sprung weight
x_T X co-ordinate of CG for total weight
x_U X co-ordinate of CG for unsprung weight
X Longitudinal co-ordinate axis in the direction of motion
X Vector of model variables

y_i Y co-ordinate of spring bottom point for i = 1 … 4
y_A Y co-ordinate for midpoint A between LF and RF springs
y_{Ae} Y co-ordinate for Aerodynamic load
y_B Y co-ordinate for midpoint B between RF and RR springs
y_{Ba} Y co-ordinate for Ballast load
y_C Y co-ordinate for midpoint C between LR and RR springs
y_D Y co-ordinate for midpoint D between LF and LR springs
y_{Dr} Y co-ordinate for Driver load
y_{Fu} Y co-ordinate for Fuel load
y_{ri} Y co-ordinate of ride point for i = 1…4
y_{Ri} Y co-ordinate for roll bar bottom point i = 1 … 4
y_{Si} Y co-ordinate for roll shock bottom point i = 1 … 4
y_S Y co-ordinate of CG for sprung weight
y_T Y co-ordinate of CG for total weight
y_U Y co-ordinate of CG for unsprung weight
Y Lateral co-ordinate axis in a direction perpendicular to motion

z_{ri} Z co-ordinate of ride point i = 1 … 4. Ride
z_A Z co-ordinate for midpoint A between LF and RF springs
z_{Ae} Z co-ordinate for Aerodynamic load
z_B Z co-ordinate for midpoint B between RF and RR springs
z_{Ba} Z co-ordinate for Ballast load
z_C Z co-ordinate for midpoint C between LR and RR springs
z_D Z co-ordinate for midpoint D between LF and LR springs
z_{Dr} Z co-ordinate for Driver load
z_{Fu} Z co-ordinate for Fuel load
z_{Ri} Z co-ordinate for roll bar bottom point i = 1 … 4
z_{Si} Z co-ordinate for shock bottom point i = 1 … 4
z_S Z co-ordinate of CG for sprung weight
z_T Z co-ordinate of CG for total weight
z_U Z co-ordinate of CG for unsprung weight
Z Vertical co-ordinate axis

The following formula is used as part of the Wan31 **Roll Bar Blades** worksheet.

Roll Bar rate with blades at variable angle

$$I_S = \frac{bH^3}{12}\frac{1}{F} \qquad I_W = \frac{Hb^3}{12}\frac{1}{F}$$

$$F = \left[\frac{H-h}{H}\right]^3\left[\ln\left(\frac{H}{h}\right) - 2\left(\frac{H}{h}\right) + \frac{1}{2}\left(\frac{H}{h}\right)^2 + \frac{3}{2}\right]$$

$$I_\alpha = \left(\frac{I_S + I_W}{2}\right) + \left(\frac{I_S - I_W}{2}\right)\text{Cos } 2\alpha$$

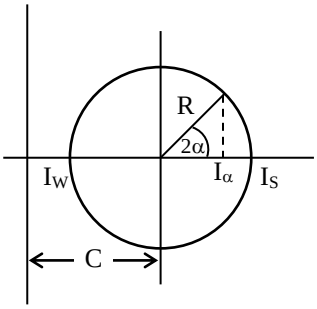
$$C = \left(\frac{I_S + I_W}{2}\right) \qquad R = \left(\frac{I_S - I_W}{2}\right)$$

$$I_\alpha = C + R\,\text{Cos } 2\alpha$$

$$d_M = \frac{Pl^3}{EI_\alpha} \qquad d_V = \frac{1.2Pl}{Gb(H-h)}\ln\left(\frac{H}{h}\right)$$

$$K_B = \frac{P}{d_M + d_V} \qquad K_T = \frac{\pi G\left(d_o^4 - d_i^4\right)}{32Ll^2}$$

$$K_{RB} = \frac{K_B\,K_T}{K_B + K_T}$$



Mohr's circle for moment of inertia

I_S = Moment of inertia of the blade, about its *strong* axis, at the root.
I_W = Moment of inertia of the blade, about its *weak* axis, at the root.
b = Thickness of the blade
H = Depth of the blade at the root
h = Depth of the blade at the tip
F = Reduction factor to account for taper
α = Angle of the blade. [$\alpha = 0^o \rightarrow$Strong; $\alpha = 90^o \rightarrow$Weak]
I_α = Moment of inertia at an angle α
C = Centre of Mohr's circle for moment of inertia
R = Radius of Mohr's circle for moment of inertia
d_M = Blade tip deflection due to bending
d_V = Blade tip deflection due to shear
P = Force at the tip of the blade
l = Blade length
L = Torque tube length
d_o = Outside diameter of torque tube
d_i = Inside diameter of torque tube
$d_i = d_o - 2\,t$
t = Wall thickness of torque tube
E = Modulus of elasticity of blade material
G = Shear modulus of plate material
K_B = Blade stiffness (Rate)
K_T = Torque tube stiffness
K_{RB} = Roll bar stiffness (Rate)